expressions x^y and y^x. For example, $(-2)^{3/7}$ is seven-valued, and $(-2)^e$ is infinitely-many-valued, each value being a complex number. Accordingly, we shall say that (a, b) is a point of the locus (1) if by proper choice from the values of a^b and b^a we have $a^b = b^a$.

Obviously the equation is satisfied if $y = x$, except at (0, 0). If $x = 0, y = 0$, we have the symbol 0^0, which is customarily considered meaningless (not many-valued), except as it is associated with the limit of a function. If we take $y = x$ in (1) and let $x = 0$ we obtain the symbolic equation $0^0 = 0^0$, and in this sense the point (0, 0) is on the locus of (1) also.

Equation (1) is equivalent to

$$e^{y \log x} = e^{x \log y}.$$

The logarithmic function is many-valued; this is shown explicitly by writing our equation

$$e^{y[\text{Log}\,|x|+i(\theta+2p\pi)]} = e^{x[\text{Log}\,|y|+i(\phi+2q\pi)]}.$$

Here $\text{Log}\,|x|$ is the single-valued real logarithm of the absolute value of x, and $\text{Log}\,|y|$ of y; p and q are positive or negative integers or zero, and θ and ϕ are zero or π according as x and y, respectively, are positive or negative numbers. These two exponentials are equal if and only if the exponents differ by $2n\pi i$, n an integer. Hence (1) is equivalent to

$$y\,\text{Log}\,|x| + i(\theta + 2p\pi)y = x\,\text{Log}\,|y| + i(\phi + 2q\pi)x + 2n\pi i.$$

For real solutions we must have simultaneously

$$y\,\text{Log}\,|x| = x\,\text{Log}\,|y|, \tag{5a}$$

$$(\theta + 2p\pi)y = (\phi + 2q\pi)x + 2n\pi. \tag{5b}$$

The graph of (5a) is obtained by the method used in discussing (2). The figures corresponding to Fig. 1 and Fig. 2 are given in Fig. 3 and Fig. 4. The

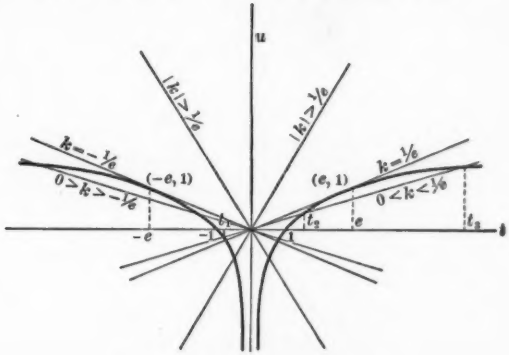


FIG. 3. $u = \log|t|$ and $u = kt$.

points of the locus in Fig. 4 which satisfy (5b) for some choice of p, q and n constitute the graph of (1). We have already seen that the line $y = x$ and the curve C_1 are parts of the locus; it remains to consider C_2, C_3 and C_4.

In the second quadrant $\theta = \pi$, $\phi = 0$, and (3b) becomes

$$(2p + 1)y = 2qx + 2n. \tag{7}$$

Not every point of C_2 is on $x^y = y^x$, for p, q, n cannot be chosen so that (7) is satisfied by $(-1, 1)$, since otherwise we would have an odd number equal to

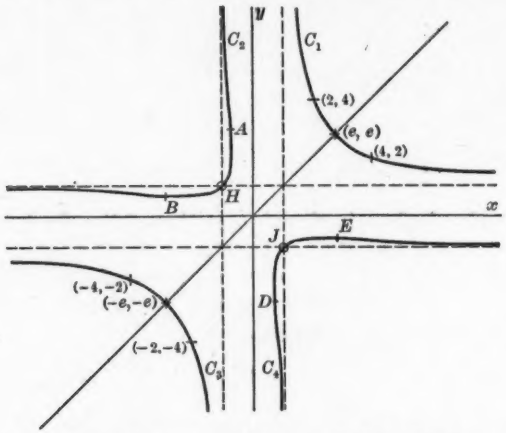


FIG. 4. Locus $y \log |x| = x \log |y|$. Includes locus $x^y = y^x$.

Special points: $A(-.7569\cdots, e)$; $B(-e, .7569\cdots)$; $D(.7569\cdots, -e)$; $E(e, -.7569\cdots)$; $H(-1, 1)$; $J(1, -1)$. B is a minimum and E a maximum of the locus.

an even number. *But the points of C_2 on $x^y = y^x$ in the vicinity of any point of C_2 form a dense point aggregate.*[1] To prove this we let (a, b) be any point on C_2 and show that it is a limit point of a set of intersections of (7) with C_2. We consider only the lines for which $q = 0$,

$$y = \frac{2n}{2p + 1},$$

which are parallel to the x-axis. It is then clearly sufficient to show that b is a lower limit point of a set of numbers of the form $2n/(2p + 1)$. Such a set is formed by taking in succession $p = 1, 2, 3, \cdots$, and choosing n so that

$$b(2p + 1) < 2n \leqq b(2p + 1) + 2. \tag{8}$$

Since the extremes of these inequalities differ by 2, there is just one integer $2n$ satisfying (8). And since (8) may be written

$$0 < \frac{2n}{2p + 1} - b \leqq \frac{2}{2p + 1}$$

[1] PIERPONT, Theory of Functions of a Real Variable, I, p. 167.

it is seen that in this set are an infinity of numbers greater than b and that they approach b as a limit as p increases indefinitely.

In the fourth quadrant the situation is precisely similar to that in the second.

In the third quadrant $\theta = \pi$, $\phi = \pi$, and (5b) becomes

$$(2p + 1)y = (2q + 1)x + 2n. \tag{9}$$

Hence the points of C_3 on $x^y = y^x$ in the vicinity of any point of C_3 form a dense point aggregate. For in this case we need consider only the lines when $q = p$,

$$y = x + 2n/(2p + 1),$$

which are parallel and have intercepts which are dense on the y-axis. Since C_3 is nowhere parallel to these lines the conclusion is obvious.

The points $(-2, -4)$ and $(-4, -2)$, in particular, lie on $x^y = y^x$.

Conclusion.—Fig. 4 is the graph of $\frac{\text{Log}\,|x|}{x} = \frac{\text{Log}\,|y|}{y}$. It contains all points of the graph of $x^y = y^x$; also it contains other points. Every point on the line $y = x$ and every point on the curve C_1 lies on $x^y = y^x$; the points H and J are not on the locus, but there is an everywhere dense set of points on C_1, C_2, C_3 lying on the locus. I do not know whether there are points on C_3 which are not on $x^y = y^x$, or how many points on C_2 and C_4 are not on $x^y = y^x$.

CONCERNING ROULETTES.

By GOLDIE HORTON, University of Texas.

I. If one curve rolls on another the curve traced by any point in the plane of the rolling curve is called a roulette. The rolling curve is called the moving centrode M and the fixed curve is called the fixed centrode F. To safely apply the methods of infinitesimals we shall suppose that the functions employed in defining the moving and fixed centrodes have continuous first and second derivatives.

(1) As M rolls on F, a line l in the plane of M and a point P on l go into a line l' and a point P' on l', and the point of contact T of M and F moves to T'. The range P on l is congruent to the range P' on l' and hence the pencil PT is projective with the pencil $P'T'$. Since the limiting position of the intersection of PT and $P'T'$ as P' moves back to P is the center of curvature at P of the roulette traced by P, we have

THEOREM 1. *The locus of the centers of curvature of the elements described simultaneously by all the points of a line in the plane of the moving centrode, for an infinitesimal movement, is a conic tangent to the moving centrode and also to the fixed centrode at the instantaneous center of rotation.*

Bresse attributes this theorem to Rivals.[1] It was Mannheim, however, who

[1] See *Journal de l'École Polytechnique*, Cahier 35, p. 112.

first wrote out a proof.[1] He proves it by means of projective properties after using a formula of Savary to prove the equality of certain angles.

(2) In case the moving centrode is a conic and the tracing point P is on the conic, the pencils PT and $P'T'$ are again projective. Hence

THEOREM 2. *If the moving centrode is a conic, the locus of the centers of curvature of the elements described simultaneously by the points of the conic, for an infinitesimal movement, is a conic tangent to the rolling conic and to the fixed centrode at the instantaneous center of rotation.*

This proof of this theorem is given by Mannheim in the paper above referred to.

(3) In case the moving centrode is a circle and the tracing point P is on the circle, the angle between any two rays of the pencil PT equals the angle between the corresponding rays of the pencil $P'T'$, and we have

THEOREM 3. *If the moving centrode is a circle, the locus of the centers of curvature of the elements described simultaneously by the points of the circle, for an infinitesimal movement, is a* CIRCLE *tangent to the rolling circle and to the fixed centrode at the instantaneous center of rotation.*

So far as we know this theorem has not been stated before. It is the purpose of this paper to show its importance in the theory of roulettes.

II. In case both centrodes are circles (we shall note in (6) that they can always be so regarded) there exists a very simple relation between their radii and the maximum radius of curvature of the corresponding epi- or hypocycloid. We now derive this relation and show that from it and Theorem 3 there follows a simple proof of Savary's elegant construction of the center of curvature at any point of a roulette.

(1) THEOREM 4. *Let ρ be the radius of the fixed circle, r that of the rolling circle, and R the radius of curvature of the point P on the rolling circle where the diameter through the rolling point cuts the circle on the opposite side. Then for the epicycloid*

$$\frac{\rho}{\rho + 2r} = \frac{R - 2r}{2r},$$

and for the hypocycloid

$$\frac{\rho}{\rho - 2r} = \frac{R - 2r}{2r}.$$

To establish the first relation consider the arc δ traced by P as the moving circle rolls over an arc δ' of the fixed circle. First, the arc δ is the product of $2r$ and the angle of rotation, which is the sum of the angles subtended by δ' at the centers of the two circles, that is, $\delta'/r + \delta'/\rho$; second, the arc δ is the product of R and the angle between successive normals to the roulette, which is $\delta'/(R - 2r)$. This double expression gives

$$\frac{\rho}{\rho + 2r} = \frac{R - 2r}{2r}.$$

[1] Mannheim, *Construction of Centers of Curvature, ibid.*, Cahier 37, p. 187.

For the hypocycloid ρ changes sign, and the relation becomes

$$\frac{\rho}{\rho - 2r} = \frac{R - 2r}{2r}.$$

(2) In view of Theorem 3 it follows, in either of the above cases, that the circle, which is the locus of the centers of curvature of the elements described simultaneously by the points of the rolling circle, for an infinitesimal movement, is of diameter $R - 2r$.

(3) Theorems 3 and 4 give the following construction for the center of curvature of the epicycloid or hypocycloid corresponding to any point.

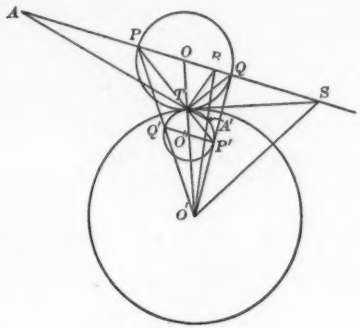


FIG. 1.

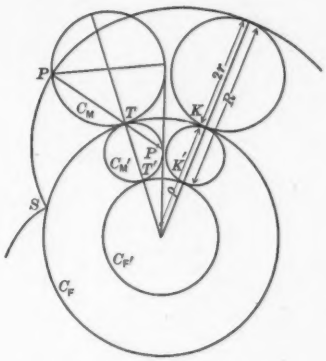


FIG. 2.

Consider the epicycloid. Let P be a point on the rolling circle (see Fig. 1). Draw PT, TQ perpendicular to PT, $O'Q$ and thus determine P'. Determine Q' similarly. Now P' is the center of curvature of the element at P of the epicycloid traced by P, and Q' is the center of curvature of the element at Q of the epicycloid traced simultaneously by Q. That is, P' and Q' are on the circle which is the locus of the centers of curvature of the elements described simultaneously by the points of the moving circle, for an infinitely small movement, and which by (2) above has for diameter $R - 2r$. For, from elementary geometry, O'' is the center of the circle described on $P'Q'$ as diameter. Also

$$\frac{P'Q'}{PQ} = \frac{\rho}{2r + \rho},$$

or,

$$P'Q' = \frac{2\rho r}{2r + \rho},$$

which by Theorem 4 is $R - 2r$; and this proves the construction.

That this construction holds for the hypocycloid may be proved similarly.

(4) We now prove that the center of curvature of the element of a roulette

described by any point A in the plane of the rolling circle is given by the following construction due to Savary. The accompanying figure shows the circles convex to each other. A proof similar to the following shows that the construction holds in the contrary case.

Draw AT, TB (see Fig. 1) perpendicular to AT, BO' and thus determine A', which we prove is the center of curvature of the element of the roulette described, by A for an infinitesimal movement. By Theorem 1 the centers of curvature of the elements of the roulettes traced by the points of the line AO, which we shall call l, lie on a conic touching the fixed and rolling circles at their point of contact T. By (3) above P' and Q' are on this conic; O' is evidently the center of curvature of every point of the roulette traced by O. The conic is therefore determined by P', Q', O' and the point of tangency T. Since the pencils TA and TB are congruent, the ranges A and B on l are projective. Then the pencils (1) and (1′) are projective and hence corresponding rays intersect on a conic. This is the conic already determined for TP and $O'Q$, TO and $O'S$, TQ and $O'P$, and the double rays TS and $O'T$ are corresponding rays of the pencils (1) and (1′).

The construction of the preceding paragraph is a special case of this.

(5) It follows from (4) that for any M and F, that is, for any definition of the movement of the points in the plane of M, the locus of the centers of curvature of the elements described simultaneously by the points on the line at infinity, for an infinitesimal movement, is a circle tangent to the fixed and moving centrodes at their point of contact, for in that case the corresponding pencils are congruent.

If M is a line tangent to the fixed centrode the locus of the centers of curvature of the elements described simultaneously by the points on the line at infinity is a circle having for diameter the radius of curvature of F at the instantaneous center of rotation.

This property is given by Mannheim in the paper already referred to.

(6) It is to be noticed that the above construction gives the center of curvature of the element described by any point in the plane of the moving centrode in any case in which one can construct the centers of curvature of the fixed and moving centrodes at their point of contact.

(7) As an application of Theorem 4 we shall prove the following well-known theorem.

THEOREM 5. *The evolute of any epicycloid is a similar epicycloid.*

By II (2) the centers of curvature of the elements described simultaneously by the points of the moving circle C_M of radius r, for an infinitesimal movement (see Fig. 2), lie on a circle $C_{M'}$ of radius $\frac{1}{2}(R - 2r)$, which by Theorem 4 may be written $\rho r/(2r + \rho)$. The ratio of the radius of $C_{M'}$ to that of C_M is therefore $\rho/(2r + \rho)$. The locus of the centers of curvature of the high points of C_M as it rolls is clearly a circle $C_{F'}$ concentric with the fixed circle C_F and is of radius $\rho - (R - 2r)$, which may be written $\rho^2/(2r + \rho)$. Then the ratio of the radius of $C_{F'}$ to that of C_F is $\rho/(2r + \rho)$. From elementary geometry we may write

$$\text{arc } TP' = \frac{\rho}{2r+\rho}\text{arc } PT = \text{arc } ST,$$

$$\text{arc } P'T' = \text{arc } TP'T' - \text{arc } TP' = \frac{\rho}{2r+\rho}\text{arc } SK^* - \frac{\rho}{2r+\rho}\text{arc } PT$$

$$= \frac{\rho}{2r+\rho}\text{arc } TK = \text{arc } T'K'.$$

This proves that as C_M rolls on C_P, $C_{M'}$ rolls on $C_{P'}$, that is, as P traces an epicycloid so does P'. Now the ratio of the radii of $C_{M'}$ and $C_{P'}$ is

$$\frac{\rho r}{2r+\rho} \div \frac{\rho^2}{2r+\rho} = \frac{r}{\rho},$$

which is the ratio of the radii of C_M and C_P. Therefore the epicycloid traced by P is similar to that traced by P'.

AN ELEMENTARY THEORY OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

By EDWARD V. HUNTINGTON, Harvard University.

In most textbooks on the calculus, the proofs of the formulas for differentiating the logarithmic and exponential functions are either confessedly incomplete, or are made to depend on a preliminary study of the complicated function, $y = \lim_{x=\infty}(1 + 1/x)^x$. This function represents one of the most difficult of the indeterminate forms, the study of which would seem more properly to come late in the course, instead of at the beginning. Moreover, the usual treatment passes over altogether too lightly the questions connected with the existence and meaning of the function a^x for irrational values of x—questions which the student can hardly be supposed to have solved satisfactorily in his previous course in algebra.

The present paper is an attempt to develop the theory of logarithms and exponents, including existence theorems and rules for differentiation, in a new way, which it is hoped will prove not only rigorous but teachable. The discussion is confined to the case of the real variable, and *no knowledge of algebra beyond positive integral exponents is pre-supposed.*

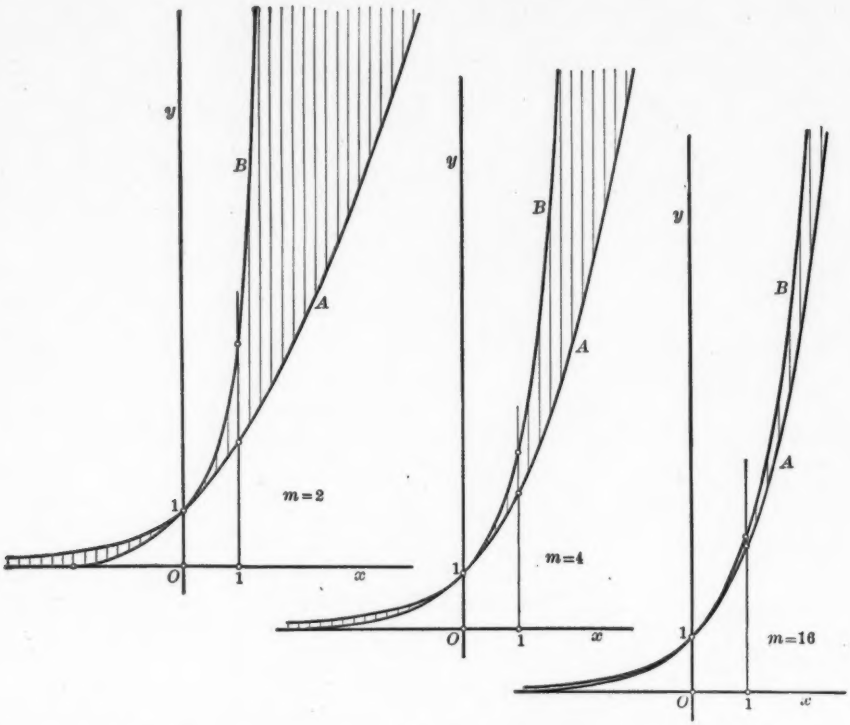
Definition of the Exponential Curve. Let us begin by supposing that in the process of plotting a variety of different curves, some one hit upon the idea of plotting the family of curves representing the following simple algebraic functions:

$$(A) \qquad (1 + x/2)^2,\ (1 + x/4)^4,\ (1 + x/16)^{16},\ \cdots (1 + x/m)^m,\ \cdots$$

$$(B) \qquad \frac{1}{(1 - x/2)^2},\ \frac{1}{(1 - x/4)^4},\ \frac{1}{(1 - x/16)^{16}},\ \cdots \frac{1}{(1 - x/m)^m},\ \cdots,$$

* Since SK is equal to a semicircumference of C_M.

where m runs through the sequence of numbers 2, 4, 16, 256, $\cdots$, each of which is the square of the preceding. It is to be understood, however, that only those values of x are considered which lie between $-m$ and $+m$. Under this restriction, the quantities in parentheses will always be positive. A few of these curves are shown in the accompanying figure.



Each of these functions has the value 1 when $x = 0$.

Fixing now our attention upon any given value of x (not zero), we proceed to prove the following facts, in corroboration of what is apparent from the figure.

(a) For any given value of x the "A" curves in the figure (beginning with the first value of m which is $> |\, x\,|$) form an ascending sequence, while

(b) the "B" curves form a descending sequence. Moreover,

(c) each "A" value is less than the corresponding "B" value, and

(d) the difference between corresponding values of the "B" and "A" curves approaches zero when m increases.

Hence, for a given value of x, the "A" sequence and the "B" sequence converge toward a common limit; and this limit is what we shall take as the definition

of the *exponential of* x, denoted by $\exp x$. The curve for $y = \exp x$ forms a boundary line between the "A" curves below it and the "B" curves above it.

Proofs of Statements (a)–(d). In order to prove the foregoing statements, (a)–(d), we first establish the following inequality:

$$(1 + d)^m > 1 + md, \tag{1}$$

where m is a positive integer and d is positive or lies between -1 and 0. The proof consists of two parts.

First, when d is positive, the inequality is obvious from the binomial theorem for positive integral exponents.

Secondly, when d lies between -1 and 0, let $b = 1 + d$ (so that $0 < b < 1$) and note that

$$1 - b^m = (1 + b + b^2 + \cdots + b^{m-1})(1 - b).$$

Hence, since $0 < b < 1$,

$$1 - b^m < (1 + 1 + 1 + \cdots + 1)(1 - b).$$

Therefore, $b^m > 1 - m(1 - b)$, or $(1 + d)^m > 1 + md$.

Having established this inequality, we now prove (a)–(d) as follows:

(a) By (1), $(1 + x/m^2)^m > 1 + x/m$ since $|x/m| < 1$ and therefore $|x/m^2| < 1$. Hence, $(1 + x/m^2)^{m^2} > (1 + x/m)^m$; that is, each of the "$A$" values is greater than the preceding.

(b) By (1), $(1 - x/m^2)^m > 1 - x/m$, since $|x/m| < 1$ and therefore $|x/m^2| < 1$. Hence, $(1 - x/m^2)^{m^2} > (1 - x/m)^m$ or $1/(1 - x/m^2)^{m^2} < 1/(1 - x/m)^m$; that is, each of the "$B$" values is less than the preceding.

(c) To prove (c), suppose that $(1 + x/m)^m$ were greater than $1/(1 - x/m)^m$; then we should have $(1 - x^2/m^2)^m$ greater than 1, which is impossible, since $x^2/m^2 < 1$. Therefore, $(1 + x/m)^m < 1/(1 - x/m)^m$.

(d) To prove (d), note that the difference in question is

$$\frac{1}{(1 - x/m)^m} - (1 + x/m)^m = \left[\frac{1}{(1 - x/m)^m}\right][1 - (1 - x^2/m^2)^m].$$

By (b), as soon as m is large enough to make $|x/m| < 1$ the first factor decreases. But by (1), as soon as m is large enough to make $x^2/m^2 < 1$, we have $(1 - x^2/m^2)^m > 1 - x^2/m$, so that from this point on, the second factor is less than x^2/m. Hence, the second factor approaches zero, while the first factor remains less than a finite quantity; therefore, their product approaches zero.

We have thus established the existence of the function $\exp x$ for every real value of x, positive, negative, or zero. In particular $\exp 0 = 1$, and $\exp 1 = 2.718 \cdots$. This latter value can be computed to any desired degree of approximation by evaluating either $(1 + 1/m)^m$, or $1/(1 - 1/m)^m$, for larger and larger values of m, and is here taken as the definition of e:

$$e = \exp 1 = 2.718 \cdots .$$

Fundamental Properties of the Exponential Function. From this definition of $\exp x$ we now proceed to deduce the fundamental theorem:

$$\exp (x + y) = (\exp x)(\exp y).$$

To prove this we show that the ratio $(\exp x)(\exp y)/\exp (x + y) = 1$, as follows:

First, using the "A" approximations, $(\exp x)(\exp y)/\exp (x + y)$ is the limit of

$$\frac{[1 + x/m]^m[1 + y/m]^m}{[1 + (x + y)/m]^m}$$

as m runs through its sequence of increasing values. This expression reduces to $\left[1 + \frac{1}{m} \cdot \frac{xy}{m + (x + y)}\right]^m$, which, by (1), is $> 1 + \frac{xy}{m + (x + y)}$ when m is sufficiently large; hence, the limit in question must be $\geqq 1$.

Secondly, using the "B" approximations, $(\exp x)(\exp y)/\exp (x + y)$ is the limit of

$$\frac{1/[1 - x/m]^m \cdot 1/[1 - y/m]^m}{1/[1 - (x + y)/m]^m}.$$

This expression reduces to $1 \div \left[1 + \frac{1}{m} \cdot \frac{xy}{m - (x + y)}\right]^m$, which, by (1), is $< 1 \div \left[1 + \frac{xy}{m - (x + y)}\right]$; hence, the limit in question must be $\leqq 1$.

Comparing these two results, we see that the limit must be precisely 1. That is, $(\exp x)(\exp y)/\exp (x + y) = 1$, or $\exp (x + y) = (\exp x)(\exp y)$.

From this fundamental addition theorem, the following theorems are easily derived:

$$\exp 0 = 1, \qquad \exp (-x) = \frac{1}{\exp x}, \qquad \exp (x - y) = \frac{\exp x}{\exp y};$$

also,

$$\exp n = e^n, \qquad \exp \frac{1}{n} = \sqrt[n]{e} = e^{1/n}, \qquad \exp \frac{m}{n} = \sqrt[n]{e^m} = e^{m/n},$$

where m and n are positive integers.

These last results show that whenever x is a rational number, $\exp x$ is precisely equal to what would be called in algebra e^x, that is, the xth power of $2.718\cdots$. On account of our familiarity with the exponential notation, it is usually convenient to write e^x instead of $\exp x$, even when x is not a rational number; we may therefore use the equation,

$$e^x = \exp x$$

as the definition of e^x in the general case.

In this notation, the fundamental theorem assumes the familiar form

$$e^{x+y} = e^x e^y.$$

The notation $\exp x$ may be retained, however, in cases where x is a long expression which cannot conveniently be printed as an exponent.

It is easily seen from the figure that e^x is > 1 when x is positive; hence, whenever $x > y$ (say $x = y + c$), we have $e^x = e^{y+c} = e^y e^c > e^y$, so that e^x is an increasing function.

Moreover, as x increases positively, e^x increases indefinitely, and as x increases negatively, e^x approaches zero.

Derivative of e^x. To prove the existence of the derivative of the function $y = e^x$ and to find its value at any point we form the difference-quotient in the usual way as follows:

$$\frac{\Delta y}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x} = \frac{e^x e^{\Delta x} - e^x}{\Delta x} = e^x \cdot \left[\frac{e^{\Delta x} - 1}{\Delta x}\right].$$

The first factor does not depend on Δx. To find the limit of the second factor, we proceed as follows: By (a) and (b), and (1), when Δx is small enough to make $-1 < \Delta x < 0$ or $0 < \Delta x < 1$, we have

$$1 + \Delta x < \left(1 + \frac{\Delta x}{m}\right)^m < e^{\Delta x} < 1 \div \left(1 - \frac{\Delta x}{m}\right)^m < \frac{1}{1 - \Delta x},$$

and hence after subtracting 1 and dividing through by Δx,

$$1 < (e^{\Delta x} - 1)/\Delta x < 1/(1 - \Delta x), \text{ if } \Delta x \text{ is positive,}$$

or

$$1 > (e^{\Delta x} - 1)/\Delta x > 1/(1 - \Delta x), \text{ if } \Delta x \text{ is negative.}$$

The left-hand member of this inequality is constantly 1, and the right-hand member approaches 1 as a limit, as Δx approaches zero through any sequence of values whatever; hence the limit of the middle member must also be 1; that is,

$$\lim_{\Delta x = 0} \left[\frac{e^{\Delta x} - 1}{\Delta x}\right] = 1.$$

Therefore the limit of $\Delta y/\Delta x$ is e^x; that is,

$$\frac{dy}{dx} = e^x, \qquad \text{or} \qquad d(e^x) = e^x dx.$$

The formula for differentiation is thus proved at one stroke for all real values of x, positive, negative, or zero.

Definition of Natural Logarithms. Having thus established the well-known shape of the exponential curve, and proved the existence of a tangent at every point, we can at once infer that for every positive quantity N there is a real quantity z such that $e^z = N$. This quantity is then, by definition, the natural logarithm of N:

$$z = \log_e N.$$

The familiar properties of the logarithm follow immediately from the definition, in the usual way.

Derivative of $\log_e x$. The curve for $y = \log_e x$ is the same as the curve for $y = e^x$, reflected in the 45° line of the first quadrant. Hence, the slope of the curve $y = \log_e x$ at any point $x = a$ is the reciprocal of the slope of the curve $y = e^x$ at the corresponding point, $y = a$. From this it follows immediately that

$$\left[\frac{d \log_e x}{dx}\right]_{x=a} = \frac{1}{a}.$$

Or, we may write $y = \log_e x$ in the form $e^y = x$, whence $e^y dy = dx$, or $dy = (1/e^y)dx$, or $dy = (1/x)dx$.

Definition of a^x (a positive). Finally, we define a^x (where a is positive) by means of the general equation

$$\log_e (a^x) = x \log_e a, \qquad \text{or} \qquad a^x = e^{x \log_e a}.$$

When x is a positive integer (or a positive or negative rational number) this definition of a^x reduces at once to the forms which are familiar from elementary algebra.

By means of this definition, all the usual properties of a^x follow immediately from the corresponding properties of e^x; in particular, the inverse of the function 10^x gives immediately the logarithm to the base 10 with all its properties. In terms of $\log_{10}$, the definition of a^x may be written in the form.

$$\log_{10} (a^x) = x \log_{10} a, \qquad \text{or} \qquad a^x = 10^{x \log_{10} a};$$

which is the form most convenient for numerical computation.

Conclusion. The method of presentation here suggested will be found to be very much shorter and simpler than any of the older methods that give the complete results (see, for example, Chrystal's *Algebra*, or Stolz's *Allgemeine Arithmetik*), and practically as short as many of the methods of the current textbooks, which give the results only for the rational case.

ELEMENTARY PROOF OF A THEOREM DUE TO F. MORLEY.

By TOBIAS DANTZIG, Indiana University.

In a paper read before the Columbus meeting of the American Mathematical Society, December 30, 1915, Professor H. S. White mentioned a theorem due to Professor F. Morley, and first given by him in a memoir entitled: "On Reflexive Geometry."[1] The theorem follows:

If a ring of five circles be formed, the center of each upon a fixed circle and each

[1] *Transactions of the American Mathematical Society*, Vol. 8, 1907, pp. 23–24.

circle of the ring intersecting the next on this fixed circle, the five other intersections when joined in succession will form a pentacle whose vertices lie one upon each of the five circles.[1] See Fig. 2.

Professor Morley's proof of this theorem is based on considerations of synthetic geometry. It is hoped that the very simple elementary proof here given will be of interest; especially as a few other remarkable properties of the same configuration immediately follow from the method used.

1. Let us first recall a theorem of elementary geometry which may not be very well known:

If C_1, C_2 are centers of two circles intersecting in A and B (Fig. 1) *and if the*

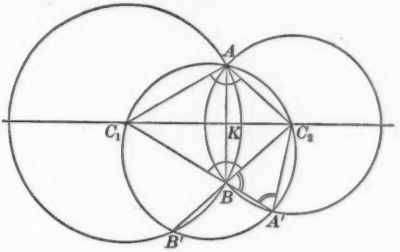


FIG. 1.

two lines C_1B and C_2B be drawn to meet the circles in A' and B', respectively, the five points A, A', B', C_1 and C_2 will lie on a circle.

For $\angle C_1AC_2 = \angle C_1BC_2$ and $\angle C_2BA' = \angle C_2A'B$; hence, we have $\angle C_1AC_2 + \angle C_2A'B = \pi$, and A' is concyclic with A, C_1 and C_2.

Conversely: *If a circle be drawn through the centers of two given circles, C_1 and C_2, and through one of their intersections say A, and if A', B' be the two other points where this circle meets the two given circles; then will the three points C_2, B, B' lie on a straight line.*

2. With the aid of this theorem Professor Morley's proposition is easily proved. Let C_1, C_2, $\cdots$ (Fig. 2) be the centers of the five circles in question all lying on the circumference O. The circles C intersect in five points A_{12}, A_{23}, $\cdots$ on the circumference O, and in five other points B_{12}, B_{23}, $\cdots$. The pentacle constructed by five joins of the points B has for vertices M_1, M_2, $\cdots$. We are to prove that the points M are on the circumferences C, say M_1 is on C_1, M_2 on C_2 and so on.

Draw C_1P and C_1Q parallel to the sides of the pentacle M_1M_4 and M_1M_3, respectively. Then

$$\angle QC_1B_{12} = \angle M_1B_{12}C_1 = \angle B_{23}B_{12}A_{23} = \angle B_{23}A_{12}A_{23},$$

and

$$\angle PC_1B_{51} = \angle M_1B_{51}C_1 = \angle B_{45}B_{51}A_{45} = \angle B_{45}A_{51}A_{45},$$

[1] This theorem can be considered as the converse of a proposition due to Auguste Miquel and generalized by Clifford. See A. Miquel. Mémoire de Géométrie, *Journal de Liouville*, Vol. X (1844), page 347. Also: Clifford, Collected Mathematical Papers, page 38.

using parallels and inscribed angles subtending equal arcs. Hence C_1P and C_1Q pass through C_4 and C_3, respectively.

Since arc $C_4C_3 = \frac{1}{2}$ arc $A_{45}A_{23}$ it follows that

$$\angle C_4C_1C_3 = \angle M_1 = \tfrac{1}{2} \angle B_{51}C_1B_{12}.$$

Now angle $B_{51}C_1B_{12}$ is central in the circle C_1 and angle M_1, subtending the same

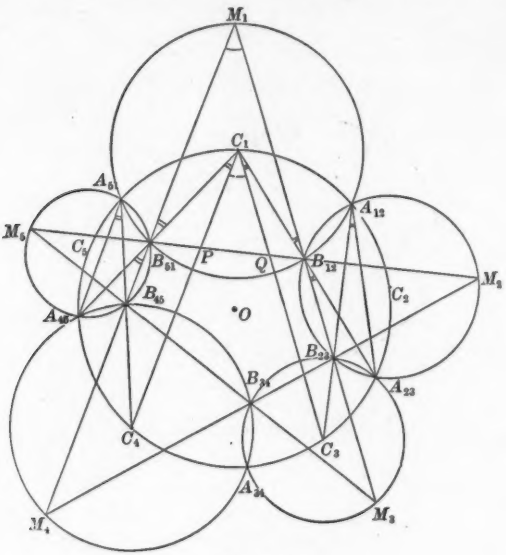


FIG. 2.

arc as the central angle and equal to one half of it, must be inscribed in the same circle. Hence the theorem.

3. By the method used the following results are readily derived:

(1) *The pentacle built on the five centers C has its sides respectively parallel to the pentacle M.*

(2) *The radical axes AB of the five circles C are respectively perpendicular to the sides of the pentacle M.*

(3) *If from any one of the points A we drop perpendiculars on the four adjacent sides of the pentacle M, the feet of the perpendiculars are on one straight line (the Simpson Line). The radical axis through A is perpendicular to the Simpson line. The five angles thus formed at A by the five perpendiculars are respectively equal to the five angles of the pentacle at M, and the Simpson lines are parallel to the sides of the pentacle M.*

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Theory of Errors and Least Squares. By LEROY D. WELD. New York: The Macmillan Company, 1916. 12mo, pp. xii + 190, with eleven figures in the text. $1.25 (weight 18 oz.).

This very useful and handy little text-book embodies the material used by the author as lecture notes during twelve years. It presents the theory of errors and least squares in such a simple and concise form as to be suitable as a text-book for a brief undergraduate course, or as a convenient reference which any research worker with a little preparation in mathematics can read in a few hours and put into immediate practice.

Though certain more or less standard developments in the method of least squares are omitted, such as Chauvenet's criterion and the elegant method by Gauss of successive substitutions for the solution of normal equations, with its useful system of notation and checks, the author covers quite adequately the essentials of the subject. In the eight chapters he discusses in order: measurement, errors, probabilities, the error equation and the principle of least squares, the adjustment of indirect observations, empirical formulas, weighted observations, and precision.

Of the 190 pages in the book about forty-five are given over to exercises; perhaps twenty more pages are devoted to well-chosen illustrations of the applications of the developments discussed; and fourteen pages are used for an appendix containing, in addition to a few of the more complicated mathematical discussions, a collection of definitions, theorems, rules and formulas for convenient reference. Only through experience could one judge whether such an extensive complement of auxiliary matter is desirable in a text and reference book in which brevity seems to have been a prominent object. However, a glance at the illustrative examples and problems will show that they are drawn from various branches of science to suggest a wide range of useful application and in a small measure to exemplify the variety of ways in which ideas relating to the theory of errors and the method of least squares adapt themselves to the daily needs of the scientist in many fields of investigation. Much of the appendix also is designed to provide a convenient means of reference for the scientist who desires to apply least squares.

Although it may cause the scientist no serious inconvenience to find non-uniformity of notation in different textbooks of least squares in reference to standard quantities such as residuals, weights and probable errors, it would be an advantage if such notation could be standardized universally enough for use in headings and margins of tabulations where brevity combined with general intelligibility is demanded. Whereas Professor Weld employs the symbols ϵ and w to denote the much used quantities, probable error and weight, in eight well known textbooks which I find at hand discussing least squares in English

these quantities are designated by r and p respectively. Probably the chance of interesting investigators in the use of the method of least squares would be increased if confusion due to non-uniformity of notation could be avoided.

Every intelligent observer desires some concrete expression of the quality of his observations; the computer who has to combine the results of different series of observations should have some knowledge of their relative accuracy in order to assign to each series its proper weight; and the investigator engaged in a complicated series of experiments desires some criterion by which to estimate the relative errors of the several parts of his work and to apportion properly his care among them. Experienced judgment will go a great way, but the working method of least squares, which has been developed on the basis of experience and analysis into its present form by a succession of thinkers, presents clearly the safest means of obtaining the result of highest probability from a given set of observations and provides methods for appraising the accuracy of such a result and for expressing this accuracy vividly to others. The present volume is planned with the purpose of making the elements of the theory of errors and least squares easily attainable both by students and by research workers. It will prove useful in the class room, in reference libraries and also on the desk.

R. H. CURTISS.

Plane Trigonometry, with Tables. By A. M. HARDING and J. S. TURNER. G. P. Putnam's Sons, New York, 1915. 158 + 51 pages. $1.10 net.

It would be decidedly interesting to know how many trigonometries have been written since Nasir Eddin (or whoever it was) delivered the subject from its bondage to astronomy. It would be much more interesting to know how many authors have, more or less unconsciously, recast the material in the mould of an older form.

At present, in the writing of American trigonometries, we seem to be passing through a period which may, with some appropriateness, be called a "reversion to type." The trigonometry under review, like the Kenyon and Ingold (published 1913) and the Wilczynski and Slaught (published 1914), makes the discussion and solution of triangles the first consideration and admits no diversion until this problem has been completely solved. In the preface we find:

"During the last few years great modifications have been made in the method of presenting plane trigonometry. Formerly the student was introduced to the trigonometric functions without any explanation of their practical utility, and spent three-fourths of his time groping in the dark with trigonometric formulas and identities. The result was that the average student found the subject repulsive.

"In accordance with the modern tendency we have introduced the ratios a few at a time, and then proceeded as soon as possible to the solution of triangles, leaving the more difficult theoretical parts of the subject until the last."

And yet the Davies' *Legendre*, almost universally used in this country for a considerable period more than fifty years ago, made the solution of triangles

the first consideration and, after having completely disposed of that, took up the general properties and relations of the trigonometric functions (functions of any angle, fundamental relations, addition and factoring formulas, etc.) under the title "Analytical Trigonometry." The same order is to be found in Loomis's *Trigonometry* (copyright 1858) of which more than 60,000 copies were printed before the first revised edition was issued.

The comprehensive and well-written textbook of Chauvenet (copyright 1850) placed the treatment of the general properties and relations of the trigonometric functions before any solutions of triangles. This arrangement was adopted by the popular textbooks of Olney (copyright 1870), Wheeler (copyright 1876), Oliver, Wait and Jones (copyright 1881) and Wells (copyright 1883).

Wentworth's trigonometry appeared in 1882 with an order of treatment that seemed to be a kind of compromise between that of Davies' *Legendre* and Chauvenet. The solution of right triangles was placed near the beginning; between that and the solution of oblique triangles was inserted, under the caption "Goniometry," a discussion of the general properties and relations of the trigonometric functions whose content was slightly greater than the "Analytic Trigonometry" which followed the solution of triangles in the Davies' *Legendre* text, and nearly equivalent to three of the chapters (II, III and IV) which preceded the solution of any triangles in the Chauvenet. Simon Newcomb's trigonometry appeared about the same time with a somewhat similar arrangement, but with more of the analytic material preceding the chapter on right triangles.

Wentworth's trigonometry may not have been the first of its type but its great success undoubtedly accounts for the fact that one can readily name ten or fifteen trigonometries published since 1890 which have adopted this hybrid order of placing the "analytical trigonometry" or "trigonometric analysis" between the discussion of right triangles and the discussion of oblique triangles. The tendency at present, of which the book under review is representative, seems to be to put the treatment of oblique triangles back to its original position.

The introductory chapter of the book under review is distinctive. At least, the reviewer knows of no other trigonometry beginning with a similar chapter intended to serve as a connecting link between plane geometry and plane trigonometry. This chapter consists of 13 pages devoted to the measurement of lines, areas and angles (in the sexagesimal system only), the proofs of five theorems on similar triangles, and 19 exercises. At least a part of this material would not be new to a college freshman, but the authors seem to take no cognizance of this fact. The reviewer wonders if students reading the chapter might not be puzzled to understand why proofs are given of some of the theorems they have already proved in elementary geometry and why, at the same time, the validity of other theorems is assumed in making these proofs.

A distinctive feature of Chapter II is that the trigonometric functions are introduced two at a time. After defining the sine and cosine of an angle, several exercises are introduced to fix these concepts before introducing the tangent and

cotangent. Similar sets of exercises are given after each of the other two pairs. The second chapter also includes eight of the fundamental relations of the functions, the functions of the special angles 30°, 45°, 60°, and of complementary angles, and an explanation of the use of a table of natural functions. The explanation of the table is well written and the reviewer believes its introduction at this place is a better plan than to put it back with the tables.

Chapter III treats of right-angled triangles and the experience of many teachers attests that the authors are wise in making no use of logarithms throughout the chapter.

The next five chapters (IV to VIII inclusive) treat of logarithms, oblique triangles, trigonometric functions of any angle, trigonometric functions of two or more angles, circular measure, inverse functions and trigonometric equations, in the order named. While there are some slight changes in these chapters which may be considered improvements, the reviewer has not discovered anything startlingly different from other customary treatments.

The closing chapter (IX) treats of some applications of trigonometry to geometry. This chapter again suggests the older (as well as various English) trigonometries, since the indebtedness to Chauvenet and Loney, acknowledged in the preface, is here most in evidence. Such a chapter, exhibiting the value and power of trigonometry in dealing with old forms, will probably contribute to increased interest on the part of the student. About 600 exercises are included and the answers to part of them are collected after Chapter IX.

The mechanical make-up of the book leaves much to be desired. The figures (cf. Figs. 45, 53) and pages of formulas (cf. pp. 98–100) suffer somewhat in comparison with others which readily come to mind. The use of disproportionately small exponents (cf. pp. 112–113), roman θ and italic ϕ in the same formula (cf. pp. 108, 122) and of a clumsy sign for $\neq$ (cf. pp. 69, 114) suggests lack of the best equipment for mathematical printing.

While one may not agree entirely with the selection or order of subject-matter, one feels immediately that this book was written by men who know something of the processes through which a freshman's mind passes in studying trigonometry. One feels confidence that the book is usable—that it "will teach," as it is sometimes expressed.

A notable omission, the absence of any use of coördinates, also intimates a turning toward older methods. But, after all, the biologists assure us that "reversion to type" is a true test of species.

U. G. MITCHELL.

UNIVERSITY OF KANSAS.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

463. Proposed by H. O. HANSON, East Elmhurst, N. Y.

Find the sum of the series

$$\binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n},$$

where $\binom{n}{r}$ denotes the coefficient of x^r in the expansion of $(1+x)^n$.

464. Proposed by GEORGE Y. SOSNOW, Newark, New Jersey.

Find the general term and the sum of n terms of the series 1, 4, 15, 56, $\cdots$, where

$$U_n = 4U_{n-1} - U_{n-2}.$$

GEOMETRY.

495. Proposed by N. P. PANDYA, Sojitra, India.

A point P moves so that the quadrilateral $PBCD$ is half of a given quadrilateral $ABCD$. Find the locus of P.

496. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find all the lines such that the pairs of tangent planes to a given sphere (ellipsoid) passing through them, shall be orthogonal.

CALCULUS.

413. Proposed by OSCAR S. ADAMS, U. S. Coast and Geodetic Survey, Washington, D. C.

Determine a function of x independent of b such that $\int_b^{b+1} f(x)dx = \frac{1}{b+1}$, the real part of b being positive.

414. Proposed by C. N. SCHMALL, New York City.

Among spherical triangles having the same base and equal altitudes, show that the isosceles triangle has the greatest vertical angle.

Show that this is also true for plane triangles.

MECHANICS.

330. Proposed by PAUL CAPRON, U. S. Naval Academy.

A barker's mill operates under a head of h feet; the linear speed of the orifice is u feet per second; the speed of the water relative to the orifice is v feet per second; and the coefficient of discharge is c, so that $v^2 = c^2(2gh + u^2)$. Given that the work done by the water on the mill is $u(v-u)/g$ foot-pounds per second per pound of water used, find the values of u and v such that the water-power may be most economically used, and find what part of the power is so used.

331. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A cyclist is riding due west at a speed of 12 miles per hour, and the wind is at the time blowing from the southeast with a speed of 5½ miles per hour. If the cyclist carries a small flag, in what direction will this flag fly? At what speed would the cyclist need to ride if the flag is to fly due north?

NUMBER THEORY.

249. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A perfect number is a number which is equal to the sum of all its different divisors. In an old book on mathematics, the following method is given without proof for determining perfect numbers. The number $2^{n-1}(2^n - 1)$ is a perfect number if $2^n - 1$ is a prime number. Prove the formula.

250. Proposed by JOSEPH E. ROWE, State College, Pa.

Show by comparatively elementary means that the equation $x^{2n} + y^{2n} = z^{2n}$ is impossible of solution in positive integers x, y, z, and n, unless at least one of the integers $x, y, z \equiv 0 \pmod 3$. In particular, consider the case $n = 1$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

450. Proposed by J. E. ROWE, Pennsylvania State College.

If the four roots of the quartic equation, $A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, are so related that $B \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$, show by elementary algebra that two roots of A are real and two imaginary. Show also by means of elementary algebra that A cannot have two equal roots without having three, if the condition $B = 0$ is satisfied.

SOLUTION BY J. A. BULLARD, Worcester, Mass.

Let

$$A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 \equiv (ax^2 + 2bx + c)(a'x^2 + 2b'x + c') = 0.$$

Then

$$a_0 = a'a, \quad 2a_1 = a'b + ab', \quad 6a_2 = a'c + 4b'b + ac', \quad 2a_3 = b'c + bc' \quad \text{and} \quad a_4 = c'c;$$

whence,

$$a_0a_4 = a'ac'c,$$

$$-4a_1a_3 = -ab'^2c - a'b^2c' - a'b'bc - abb'c',$$

$$3a_2^2 = \tfrac{1}{12}(a'c)^2 + \tfrac{4}{9}(b'b)^2 + \tfrac{1}{12}(ac')^2 + \tfrac{2}{3}a'b'bc + \tfrac{2}{3}abb'c' + \tfrac{1}{6}aa'cc'.$$

Adding these equations and simplifying, we have

$$\text{(I)} \qquad B \equiv (b^2 - ac)(b'^2 - a'c') + \tfrac{1}{3}[bb' - \tfrac{1}{2}(a'c + ac')]^2.$$

If the original coefficients are real numbers then a, b, c, a', b', c' can always be taken so as to have real values. If $B = 0$ and the roots are all distinct the product of the discriminants $b^2 - ac$ and $b'^2 - a'c'$ is negative and hence one discriminant is negative and the other positive. Thus two of the four roots of $A = 0$ are the real and unequal roots of one quadratic and the other two roots are the imaginary roots of the second quadratic.

If $B = 0$ and two roots are equal let us assume them to be the roots of $ax^2 + 2bx + c = 0$, that is, $b^2 - ac = 0$. Then from (I) it follows that $bb' - \tfrac{1}{2}(a'c + ac') = 0$. Solving the quadratics and substituting from the relations just stated, we find the roots to be $-\frac{b}{a}, -\frac{b}{a}, -\frac{b}{a}, -\frac{bc'}{a'c}$. Thus, if $B = 0$ and two roots are equal, three roots must be equal.

A further examination of (I) shows that

If $B < 0$, two roots are imaginary and two are real and unequal.

If $B = 0$, two roots are imaginary,and two are real and unequal, or three are equal, or all are equal.

If $B > 0$, two are imaginary and two are equal, or all are imaginary, or all are real. In this case we may have two double roots.

Also solved by GEORGE W. HARTWELL and the PROPOSER.

451. Proposed by H. S. UHLER, Yale University.

Prove that

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

SOLUTION BY W. E. MILNE, Bowdoin College.

Applying successively the formula $\sin 2\theta = 2\cos\theta\sin\theta$ we obtain

$$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2}, \quad = 4\cos\frac{x}{2}\cos\frac{x}{4}\sin\frac{x}{4}, \quad = 8\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\sin\frac{x}{8}, \quad \cdots,$$

and in general

$$\frac{\sin x}{2^n \sin(x/2^n)} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}. \tag{1}$$

Now $\lim_{n=\infty} 2^n \sin(x/2^n) = x$. Since equation (1) holds for every positive integral value of n, and the left-hand side approaches a limit $\frac{\sin x}{x}$ as n becomes infinite, it follows that the right-hand side also approaches a limit, and the limits are equal. Therefore,

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

Also solved by E. B. WILSON, PAUL CAPRON, HORACE OLSON, C. A. NICKLE, A. H. HOLMES, H. S. BROWN, E. H. VANCE, NORMAN ANNING, J. A. BULLARD, J. J. GINSBURG, C. K. ROBBINS, H. L. AGARD, H. R. HOWARD, O. S. ADAMS, I. A. BENNETT, J. A. CAPARO, and the PROPOSER.

GEOMETRY.

479. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the locus of the point whose polars (polar planes) with respect to two given conics (quadrics), are perpendicular to each other.

SOLUTION BY THE PROPOSER.

A. POLARS WITH RESPECT TO CONICS.

If P is a point of the required locus, its polars p_1, p_2 with respect to the two given conics (Σ_1), (Σ_2) meet the line at infinity in two points Q_1, Q_2 separated harmonically by the cyclical points $\mathfrak{I}$, $\mathfrak{I}'$ at infinity. The pole, with respect to (Σ_1), of any line passing through Q_1 lies on the polar q_1 of Q_1 with respect to (Σ_1); and the pole, with respect to (Σ_2), of any line passing through Q_2 lies on the polar q_2 of Q_2 with respect to (Σ_2). When Q_1 varies on the line at infinity, q_1 turns about the center C_1 of (Σ_1), the range (Q_1) and the pencil (q_1) being projective. Similarly for the range (Q_2) and the pencil (q_2). The points Q_1, Q_2 being a couple of conjugate elements in the involution whose double elements are the cyclical points $\mathfrak{I}$, $\mathfrak{I}'$, we have

$$(q_1\cdots) \barwedge (Q_1\cdots) \barwedge (Q_2\cdots) \barwedge (q_2\cdots)$$

Hence: *The locus of the point, whose polars, with respect to two given conics, are perpendicular to each other, is, in general, a conic passing through the centers of the given curves.*

I. Two central conics. If (Σ_1) and (Σ_2) are both central conics, let (C) be the circle having for diameter the line C_1C_2 of their centers. The couples of conjugate diameters of (Σ_1) determine an involution (I_1) of points on (C). Let O_1 denote the pole of this involution. The conic (Σ_2) gives rise to a similar involution (I_2) with O_2 as its pole. Let M be any point of (C). If M_1, M_2 are the two other points of intersection of (C) with the lines O_1M, O_2M, respectively, the point of intersection P of the diameters C_1M_1, C_2M_2 will belong to the required locus (Σ), because the polars of P with respect to (Σ_1) and (Σ_2) are parallel to the diameters C_1M and C_2M, respectively conjugate to C_1P, C_2P in the two given conics, and the two former are perpendicular to each other. This affords an expeditious way for the construction of (Σ).

The points of intersection N, N' of the circle (C) with the line O_1O_2 belong to the required locus (Σ), as is readily seen by making the point M coincide with each one of them in turn. These two points will necessarily be real, if at least one of the points O_1, O_2 lies inside of (C).

The double elements of (I_1) on (C) are the points of contact of the tangents to (C) drawn from O_1. The lines joining C_1 to these points of contact are the double elements of the involution of

conjugate diameters of (Σ_1), *i. e.*, the asymptotes of (Σ_1). Hence O_1 will lie outside of (C) if (Σ_1) is an hyperbola, and inside of (C) if it is an ellipse or a circle. Similarly for O_2.

The conic (Σ) will have a real point at infinity if the lines C_1M_1, C_2M_2 are parallel. The line C_1C_2 being a diameter of (C), the quadrilateral $C_1M_1C_2M_2$ inscribed in (C) is a rectangle, and M_1M_2 is a diameter, hence $\angle M_1MM_2 = \angle O_1MO_2 = 90°$. The point M is therefore common to (C) and to the circle having O_1O_2 for diameter. The points of intersection of these two circles are necessarily real and distinct when one of the points O_1, O_2 is inside and the other outside of (C). They may be real, imaginary, or coincident, when O_1, O_2 are both inside, or both outside of (C). Consequently: *The conic* (Σ) *is, in general, an hyperbola, if one of the given conics is an hyperbola and the other an ellipse. Otherwise* (Σ) *may be an ellipse, an hyperbola, or a parabola.*

Since the conic (Σ) passes through the four points C_1, C_2, N, N' of (C), the conic can have no other point in common with (C) and remain distinct from this circle. Therefore, if (Σ) is to be a circle distinct from (C) this is only possible, if the points of intersection of O_1O_2 with (C) are the cyclical points at infinity, *i. e.*, O_1O_2 must itself be the line at infinity, and hence the points O_1, O_2 are points at infinity. But the tangents to (C) drawn from O_1 at infinity touch (C) at two diametrically opposite points, the asymptotes of (Σ_1) are therefore real and orthogonal. Similarly for (Σ_2). Hence: *The locus of the point whose polars, with respect to two given equilateral hyperbolas, are perpendicular to each other, is, in general, a circle distinct from the circle having for diameter the line of centers of the given conics.*

The reader may prove the following statements: If an asymptote of (Σ_1) is perpendicular to an asymptote of (Σ_2), the conic (Σ) is tangent to (C) at the point of intersection X of these lines. If X coincides, say, with C_1, (C) is the osculating circle of (Σ) at this point. If C_1C_2 is an axis for each of the given conics, (Σ) has a double contact with (C) at C_1, C_2.

If O_1C_2, O_2C_1 meet on (C), to the line C_1C_2 of the pencil (q_1) at C_1 will correspond the line C_2C_1 of the pencil (q_2) at C_2, so that the line C_1C_2 will be a part of the locus (Σ), the other part being the line O_1O_2. Hence: *If the diameters conjugate in the two given conics, respectively, to the line of their centers, are perpendicular to each other, the locus degenerates into two straight lines, one of them being the line of their centers.*

The conic (Σ) will be identical with (C), if for any point M of (C) the points M_1, M_2 coincide; but then the lines O_1MM_1 and O_2MM_2 will coincide, which makes it in turn necessary for O_1, O_2 to coincide, say in O. It is evident that this last condition is also sufficient for the coincidence of (Σ) with (C). If the involutions (I_1) and (I_2) have the same pole O, the diameters C_1M, C_1M_1 of (Σ_1) which are perpendicular to any couple of conjugate diameters C_2M, C_2M_2 of (Σ_2), are themselves conjugate in (Σ_1). Consequently: *If the couples of conjugate diameters of one of the given conics are the perpendiculars dropped from its center upon the couples of conjugate diameters of the other conic, the required locus is the circle* (C). This will take place, for instance, when the two given conics are both circles, or hyperbolas with mutually perpendicular asymptotes.

If the given conics are concentric, the two projective pencils (q_1), (q_2) are superposed, and their double elements constitute the required locus. These lines may conveniently be constructed by drawing any circle (C) through the common center C of the given conics and determining the points O_1 and O_2 as above; the lines projecting from C the points of intersection of (C) with the circle having O_1O_2 for its diameter, are the required lines. They are always real when one of the given conics is an ellipse (a circle) and the other an hyperbola. When the points O_1, O_2 coincide, *i. e.*, when the given conics have the same involution of conjugate diameters, the required locus has no real points.

N. B. In the above discussion of the central conics (Σ_1), (Σ_2), only their involutions of conjugate diameters were considered. The locus (Σ) would therefore not change, if (Σ_1) would be replaced by any other conic of the pencil having the same involution of conjugate diameters as (Σ_1). Similarly for (Σ_2). For instance, an hyperbola may be replaced by any hyperbola having the same asymptotes: a circle, by any other concentric circle.

II. A parabola and a central conic. The locus (Σ) will pass through the center C_1 of (Σ_1) and through the point at infinity of the parabola (Σ_2). If q_1, q_1' are any two conjugate diameters of (Σ_1), t_2 the tangent to (Σ_2) perpendicular to q_1', the diameter q_2 of (Σ_2) conjugate to t_2 will meet q_1 in a point of (Σ). If d_1, d_1' are the two conjugate diameters of (Σ_1), of which one, say d_1', is perpendicular to the axis a_2 of (Σ_2), the point at infinity of d_1 belongs to (Σ), as may be readily seen. Hence: *The required locus is, in general, an hyperbola passing through* C_1, *whose asymptotic directions are* a_2 *and* d_1.

The two asymptotic directions of (Σ) will be perpendicular to each other, when and only when the conjugate diameter d_1 of d_1' coincides with d_1', *i. e.*, when d_1' is an asymptote of (Σ_1). Hence:

The locus will be an equilateral hyperbola, if the axis of the parabola is perpendicular to an asymptote of the other conic.

The two points at infinity of the locus (Σ) will coincide, if d_1 is parallel to a_2, *i. e.*, if d_1' is an axis of (Σ_1). Hence: *The required locus will be a parabola, if the axis of the given parabola is parallel to one of the axes of the other given conic.*

If the central conic is a circle, the locus is always a parabola.

In the construction above, if for the diameter q_2 is taken the one n_2 passing through C_1, the corresponding diameter n_1 of (Σ_1) is the tangent to (Σ) at this point. The diameter n_1 will coincide with n_2, if the conjugate directions of n_2 in (Σ_1) and (Σ_2), respectively, are perpendicular to each other. The line n_2 is then a part of (Σ). Hence: *If the directions conjugate in the two given conics to the line of their centers, are perpendicular to each other, the locus degenerates into two straight lines, one of which is their line of centers.*

The reader may discuss the case when the axis of the parabola passes through the center of the other given conic.

III. Two parabolas. The required locus passes through the points at infinity of the given parabolas (Σ_1), (Σ_2), the conic (Σ) is therefore always an hyperbola. Draw the tangent to (Σ_2) which is perpendicular to the axis of (Σ_1). The diameter of (Σ_1) which passes through the point of contact, is one of the asymptotes of (Σ). The other asymptote is found in the same way starting with (Σ_1). The hyperbola (Σ) will be equilateral, if the axes of the parabolas are perpendicular. If these axes are parallel, their common point at infinity is the center of the two superposed projective pencils (q_1), (q_2), whose double elements constitute the required locus.

B. Polar Planes with respect to Quadrics.

If P is a point of the required locus, its polar planes π_1, π_2 with respect to the two given quadrics (Σ_1), (Σ_2), meet the plane at infinity in two lines p_1, p_2, which are conjugate with respect to the imaginary circle at infinity (γ), *i. e.*, p_2 passes through the pole M of p_1 with respect to (γ). The poles, with respect to (Σ_1) of all the planes passing through p_1, are on the line m_1 which is the conjugate to p_1 with respect to (Σ_1); the poles, with respect to (Σ_2), of all the planes passing through M, are in the polar plane μ_2 of M with respect to (Σ_2). The point P is therefore the point of intersection of the line m_1 with the plane μ_2. If p varies in the plane at infinity, m_1 describes the bundle of lines (m_1) having for its center the center of (Σ_1), and the two forms (p_1), (m_1) are reciprocal; if the point M varies in the plane at infinity, μ_2 describes a bundle of planes (μ_{21}), having for its center the center of (Σ_2), and the two forms (M) and (μ_2) are reciprocal. The two plane forms (M), (p_1) being reciprocal with respect to (γ), the bundle of lines (m_1) is reciprocal to the bundle of planes (μ_2). Hence: *The locus of the point whose polar planes, with respect to two given quadrics, are perpendicular to each other, is, in general, a quadric passing through the centers of the given surfaces.*

Also solved analytically by Florence P. Lewis.

480. Proposed by Clifford N. Mills, Brookings, S. Dakota.

Of equal quadrilaterals on the same base, that which has the least perimeter must have the angles not adjacent to the base equal to each other.

Solution by Elijah Swift, University of Vermont.

We shall first prove a different theorem and then show it is equivalent to the above. "Of isoperimetric quadrilaterals on the same base, that which has the greatest area must have the angles not adjacent to the base equal to each other." Call the vertices $ABCD$, the given base being AD. The maximum quadrilateral must have $AB = BC = CD$, since, if the triangle ABC is a maximum, it is isosceles. Again, it must be inscriptible in a circle and hence is an isosceles trapezoid with angle B equal to angle C.

Now suppose we could find another quadrilateral, $AB'C'D$, having the same area as $ABCD$, but a smaller perimeter. It is clear that we can then construct a third quadrilateral, *e. g.*, by moving the vertex B' to B'', so that $AB''C'D$ has a larger area than $AB'C'D$ or $ABCD$, but the same perimeter as $ABCD$. But this is impossible. Also, there can be no other quadrilateral having the same area and perimeter. Hence, $ABCD$ has the least perimeter, for the given area, as well as the most area for a given perimeter.

CALCULUS.

397. Proposed by C. N. SCHMALL, New York City.

On the radii vectores of one loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$ as diameters, circles are described passing through the pole. Find the locus of their points of intersection, and show that the area is twice that of the loop.

Solution by Theodore Howard, New Haven, Conn.

Let ϕ be the angle between the polar axis and the radius vector of the circle described on the radius vector of the lemniscate as a diameter. Let r be the radius vector of the circle, the origin and polar axis being the same as for the lemniscate. Then the equation of the circle is $r = \rho \cos(\phi - \theta)$. Or $r^2 = \rho^2 \cos^2(\phi - \theta) = a^2 \cos 2\theta \cos^2(\phi - \theta)$, substituting from the equation of the lemniscate. In this equation, θ is the variable parameter. Taking the derivative of this equation with respect to θ, we have

$$0 = 2a^2 \cos(\phi - \theta)[\sin(\phi - \theta)\cos 2\theta - \sin 2\theta \cos(\phi - \theta)],$$
$$= 2a^2 \cos(\phi - \theta)\sin(\phi - 3\theta).$$

Hence, $\theta = \frac{\phi}{3}$, or $\theta = \phi - \frac{\pi}{2}$ an extraneous value. Substituting $\frac{\phi}{3}$ for θ in the preceding equation, we have $r^2 = a^2 \cos^3 \frac{2\phi}{3}$, the equation of the envelope. The area of one loop of the envelope is $A = \frac{a^2}{2}\int_b^{\pi} \cos^3 \frac{2\phi}{3} d\phi = 3a^2 \int_0^{\pi/4} \cos^3 2\theta d\theta = a^2$. The area of one loop of the lemniscate is

$$A' = a^2 \int_0^{\pi/4} \cos 2\theta d\theta = \frac{a^2}{2}.$$

Hence,

$$A = 2A'.$$

Also solved by C. E. Horne, Paul Capron, H. L. Agard, and Norman Anning.

398. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve

$$2\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0.$$

Solution by W. W. Beman, University of Michigan.

Assume $z = \phi(y + mx)$. The equation in m is $2m^2 - 3m - 2 = 0$. Hence,

$$z = \phi(y + 2x) + \psi(2y - x).$$

Also solved by Elijah Swift, J. L. Riley, and the Proposer.

399. Proposed by B. J. BROWN, Victor, Colorado.

A cow is tethered by a perfectly smooth rope, a slip noose in the rope being thrown over a large square post. If the cow pulls the rope taut in the direction shown in the figure, at what angle will the rope leave the post?

From Granville's *Diff. and Int. Calculus*, p. 120, Prob. 55.

Solution by H. L. Agard, Williams College.

The rope leaves the post in such a manner that the knot at the end of the noose is in front of the middle of one face of the post. Let a be the thickness of the post, φ the angle at which the rope leaves the post, b the distance from the knot to the corner of the post, and c the perpendicular distance from the knot to the post.

Then

$$b = \frac{a}{2}\sec\varphi \qquad \text{and} \qquad c = \frac{a}{2}\tan\varphi.$$

If l is the total length of the rope and x is the distance of the cow from the post,

$$x = l - 3a - 2b + c = l - 3a - a\sec\varphi + \frac{a}{2}\tan\varphi. \tag{1}$$

Taking the derivative of x with respect to φ and setting the result equal to 0, we have,

$$\frac{dx}{a\varphi}] = -a\sec\varphi\cdot\tan\varphi + \frac{a}{2}\sec^2\varphi = 0.$$

Hence, $\sec\varphi = 0$ and $\sec\varphi - 2\tan\varphi = 0$. Whence, φ from the last equation $= \pi/6$. This value of ϕ in $d^2x/d\varphi^2$ gives a negative value. Hence, for $\varphi = \pi/6$, x is a maximum.

Also solved by J. A. BULLARD.

NOTE. It seems to us, that the statement of this problem is misleading. As stated, it suggests a problem in mechanics and as such it is not a problem of maxima and minima except incidentally.

When the rope is taut, the tension in the three parts of the rope about the knot are equal and the knot (better a smooth ring) is then in stable equilibrium. From this fact, it follows that the rope leaves the post at an angle of 30°, and it no more involves the idea of maxima and minima than does the fact that a cube of homogeneous density has its center of gravity in the lowest position when a face of the cube is in contact with a horizontal plane.

However, it turns out in this problem that if the length of the rope is greater than $\frac{1}{2}(9 + 2\sqrt{3})$ times the length of a side of a right section of the post, the cow will be at a maximum distance from the side of the post, when the rope is taut.

To establish the reason why these two facts should coexist transcends the ability of the average student of elementary calculus.

The problem may be stated in definite form as follows: An inextensible weightless string of length $l > na$, where n is an integer greater than unity, is to be cut into two parts. The ends of one part of the string are fastened to the points A and B located in the same horizontal line, the distance between them being a. To one end of the other part of the string is attached a small weight while the other end of this part of the string is attached to the middle point of the suspended part, and the parts and weight are then allowed to hang freely. What angle will the string make with line AB when it is so cut that the weight is at a maximum distance from the line AB?

It may be argued in defense of the problem as given, that it leaves the student to discover for himself the particular quantity that becomes a maximum, instead of calling his attention to that quantity, as the proposed modification does, which is an argument worthy of consideration. But the statement of this problem leads the student by suggestion away from the discovery of the quantity which is to be made a maximum rather than directs him towards such discovery, which seems to us to be objectionable. EDITOR FINKEL.

MECHANICS.

314. Proposed by C. N. SCHMALL, New York City.

A rectangular box of height h, and having a plane mirror for its bottom, contains a quantity of water of unknown height x. In the lid are two small apertures distant $2a$ from each other. A ray of light entering one aperture with an angle of incidence φ, emerges, after refraction and reflection, through the other aperture. If μ be the index of refraction of water, show that the height of the water is

$$x = \frac{h\tan\varphi - a}{\tan\varphi - \dfrac{\sin\varphi}{(\mu^2 - \sin^2\varphi)^{1/2}}}.$$

SOLUTION BY FRANK IRWIN, University of California.

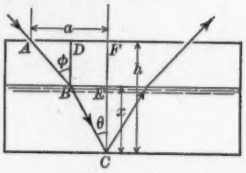
We are given, in the figure, $AF = a$, $CF = h$, $CE = x$. Then $BD = h - x$. $a = AD + BE = (h - x)\tan\varphi + x\tan\theta$. Whence $x = (h\tan\varphi - a)/(\tan\varphi - \tan\theta)$.

Now $\mu = \sin\varphi/\sin\theta$, or $\mu\sin\theta = \sin\varphi$; so that

$$\tan\theta = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}} = \frac{\sin\varphi}{\sqrt{\mu^2 - \sin^2\varphi}}.$$

Substituting this value of $\tan\theta$ in the value of x obtained above gives us

$$x = \frac{h\tan\varphi - a}{\tan\varphi - \dfrac{\sin\varphi}{(\mu^2 - \sin^2\varphi)^{1/2}}}.$$



Also solved by H. L. AGARD, J. A. CAPARO, HORACE OLSON, H. C. FEEMSTER, L. G. WELD, F. R. MORRIS, H. N. CARLETON, H. S. UHLER, and the PROPOSER.

316. Proposed by C. N. SCHMALL, New York, N. Y.

A body at rest at a point R begins to move toward a center of force F. The distance $RF = d$, and the force varies inversely as the distance. Two intermediate points in the path are P and Q, such that $FP = kd$, and $FQ = k^n d$. Show that the body will traverse the distance QP in a *maximum* of time if $k = 1/n^{1/2(n-1)}$.

SOLUTION BY H. S. UHLER, Yale University.

Since P is a point intermediate with respect to F and R, then $d > kd > 0$ or $1 > k > 0$. Again, since Q is a point between F and R, then $d > k^n d > 0$ or $1 > k^n > 0$. Hence, since k must be positive and less than unity, two possibilities arise, namely;

$$\text{(a) } n > 1, \quad \text{and} \quad \text{(b) } 1 > n > 0.$$

Case (a) $n > 1$.

Then $\overline{FQ} < \overline{FP}$. Taking F as origin and defining x as positive from F toward R, we have

$$\frac{d^2x}{dt^2} = -\frac{a}{x},$$

where a is a positive constant. Writing $v \equiv \dfrac{dx}{dt}$, and noting that $\dfrac{d^2x}{dt^2} = v\dfrac{dv}{dx}$, we obtain

$$v\,dv = -a\frac{dx}{x},$$

whence

$$\tfrac{1}{2}v^2 = c - a\log x.$$

When $x = d$, $v = 0$, therefore $c = a\log d$ and

$$\frac{dx}{dt} = -\left(2a\log\frac{d}{x}\right)^{\frac{1}{2}},$$

the negative sign indicating that the displacement decreases as time increases. Letting T denote the time required to traverse the distance from P to Q we have

$$\int_{kd}^{k^n d}\left(2a\log\frac{d}{x}\right)^{-1/2}dx = -\int_0^T dt, \qquad T = -\int_{kd}^{k^n d}\left(2a\log\frac{d}{x}\right)^{-1/2}dx.$$

Since T is to be tested for maxima and minima we now apply the ordinary method for differentiating a definite integral and obtain

$$\frac{dT}{dk} = \frac{d}{\sqrt{2a}}\left[(\log k^{-1})^{-1/2} - nk^{n-1}(\log k^{-n})^{-1/2}\right].$$

The expression within the brackets vanishes when $nk^{2(n-1)} = 1$.

It must next be shown that this value of k corresponds to a maximum of T. The second derivative of T with respect to k reduces to

$$\frac{d}{2k\sqrt{2a}}\left\{(\log k^{-1})^{-3/2} - nk^{n-1}(\log k^{-n})^{-3/2}[n + 2(n-1)(\log k^{-n})]\right\}.$$

Substituting from the condition $nk^{2(n-1)} = 1$ we find

$$\frac{d^2T}{dk^2} = -\frac{d(n-1)}{k}\left(\frac{n-1}{a\log n}\right)^{1/n}.$$

Since this result is negative ($n > 1$) a true maximum obtains.

Case (b). $1 > n > 0$.

Then $\overline{FP} < \overline{FQ}$. Proceeding as before we get

$$T' = -\int_{k=d}^{kd}\left(2a\log\frac{d}{x}\right)^{-1/n}dx,$$

where T' symbolizes the interval of time consumed in going from Q to P. Consequently $nk^{2(n-1)} = 1$ and

$$\frac{d^2T'}{dk^2} = -\frac{d(1-n)}{k}\left(\frac{1-n}{-a\log n}\right)^{1/n}.$$

Since $n - 1$ is now negative, the second derivative is also negative, and hence the condition for a maximum is again fulfilled.

Remark: In making the final reductions in case (b) attention has to be paid to the fact that $\log n$ is negative as well as $n - 1$.

Also solved by A. H. WILSON, H. N. CARLETON, H. POLISH, J. A. CAPARO, ELIJAH SWIFT, PAUL CAPRON, and the PROPOSER.

NUMBER THEORY.

209. (March, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Prove that the difference of the sixth powers of two integers cannot be the square of an integer.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

In Carmichael's Diophantine Analysis, pp. 70, 71, the impossibility of the equation $x^3 + y^3 = 2^n z^3$ is proved. (The proof is valid whether y is positive or negative.)

We are to prove the impossibility of the equation $a^6 - b^6 = c^2$. We know that if $x^2 + y^2 = z^2$, then $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$. (Loc. cit., p. 10.) We assume, of course, that a, b, c are all prime to each other; so are x, y, z also. Two cases present themselves:

(I) $$a^3 = m^2 + n^2, \quad b^3 = m^2 - n^2, \quad c = 2mn.$$

(II) $$a^3 = m^2 + n^2, \quad b^3 = 2mn, \qquad c = m^2 - n^2.$$

Case I. Of the two integers, m and n, one is odd and the other even. Also these numbers are prime to each other. Consequently $m + n$ and $m - n$ are prime to each other, and since their product is a cube, each of them must be a cube also. If we set them equal to α^3 and β^3 respectively

$$m + n = \alpha^3, \quad m - n = \beta^3, \quad a^3 = m^2 + n^2 = \frac{\alpha^6 + \beta^6}{2}, \quad \text{or} \quad \alpha^6 + \beta^6 = 2a^3,$$

the impossibility of which was stated in the first paragraph.

Case II. Assume m even; the proof will apply equally well to the other case, n even. Since $2mn$ is a cube, $2m$ and n must each of them be a cube. Setting them equal to $8\alpha^3$ and β^3, we have

$$m = 4\alpha^3, \qquad n = \beta^3, \qquad a^3 = 16\alpha^6 + \beta^6 \qquad \text{or} \qquad 2^4\alpha^6 = a^3 - \beta^6,$$

an equation which was proved to be impossible. Hence the equation $a^6 - b^6 = c^2$ cannot hold.

219. (June, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Determine whether it is possible for a polygon to have the number of its diagonals equal to a perfect fourth power.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

If the number of sides of a polygon is n, the number of its diagonals is $n(n-3)/2$, which by the conditions of the problem must be a perfect fourth power.

The fourth power of any number is divisible by 16, if the number is even, by 16 with remainder 1 if the number is odd.

We consider two cases: (I) n not divisible by 3, (II) n divisible by 3.

Case I. If n is even, $n-3$ is odd, and the two are prime to each other. Since their product is twice a fourth power, $n-3$ is a fourth power. Consequently $n-3$ is divisible by 16 with remainder 1, by n with remainder 4, and by $n/2$ with remainder 2 or 10; in neither case can $n/2$ be a fourth power, so that $n(n-3)/2$ cannot be a fourth power either.

If n is odd, it must be a fourth power by the same reasoning as before. $n-3$ has the remainder 14, and $(n-3)/2$, has the remainder 7 or 15, when divided by 16; so that $(n-3)/2$ cannot be a fourth power, nor can $n(n-3)/2$.

Case II. If n is divisible by 3, $n-3$ is also, and so is the fourth power, so that we have the equation $n(n-3) = 2\cdot 3^4\cdot k^4$. Either n or $n-3$ is divisible by 3^3 and the other one by 3. Calling the one $3^3\beta$ the other will be $3^3\beta \pm 3$, the sign depending on which is divisible by 3^3. Our equation takes the form $3^3\beta(3^3\beta \pm 1) = 2\cdot 3^4\cdot k^4$, or $\beta(3^3\beta \pm 1) = 2\cdot k^4$.

If β is odd, it must be a fourth power, and $(3^3\beta \pm 1)$ twice a fourth power. But if $\beta \equiv 1 \pmod{16}$, $3^3\beta \pm 1 \equiv 8$ or $10 \pmod{16}$, so that it could not be twice a fourth power.

If β is even, $\beta/2$ and $3^3\beta \pm 1$ must both be fourth powers. Here, again, two cases are possible: $\beta/2$ even and $\beta/2$ odd. In the latter case $\beta/2 \equiv 1 \pmod{16}$, or $\beta \equiv 2 \pmod{16}$, so that $3^3\beta \pm 1 \equiv 3$ or $1 \pmod{16}$, from which we deduce at once that only the $-$ sign is admissible. But $3^3\beta - 1$ = a fourth power, and any fourth power if divided by 3 must yield the remainder $+1$ or 0. Consequently $3^3\beta - 1$ cannot be a fourth power, if $\beta/2$ is odd.

It remains to consider the case $\beta/2$ even. Just as before, we see that $\beta/2$ and $3^3\beta \pm 1$ must be fourth powers. Calling these $16b^4$ and a^4 respectively, (since $\beta/2$ is even, it must be the fourth power of an even integer) we derive the equations $288b^4 \pm 1 = a^4$. Since $a^4 \equiv 1 \pmod{16}$. the $+$ sign only is admissible. We now proceed to show the impossibility of this last equation,

Writing it in the form $288b^4 = a^4 - 1$, we see that for the right hand side to be divisible by 32, a must be of the form $8\lambda \pm 1$. Substituting this value for a and factoring the right-hand side, the equation becomes

$$288b^4 = (64\lambda^2 \pm 16\lambda + 2)(8\lambda \pm 2)8\lambda,$$

or, after division by 32,

$$9b^4 = (32\lambda^2 \pm 8\lambda + 1)(4\lambda \pm 1)\lambda.$$

It is easy to see that the three factors on the right are prime to each other: in fact

$$(32\lambda^2 \pm 8\lambda + 1) - (4\lambda \pm 1)^2 = 16\lambda^2,$$

so that any common factor of the first two must be a factor of $16\lambda^2$. Since the product of these three is a perfect square, each of them must be a square, and we have the three equations

$$\lambda = c^2,$$

$$4\lambda^2 \pm 1 = d^2,$$

$$32\lambda^2 \pm 8\lambda + 1 = e^2.$$

From the first two of these we deduce

$$4c^2 \pm 1 = d^2,$$

which is impossible, for the squares of two integers cannot differ by unity.

As we have examined all possible cases, we conclude that no polygon can have the number of its diagonals equal to a perfect fourth power.

222. (October, 1914.) Proposed by A. H. HOLMES, Brunswick, Me.

Find rational values for m and n such that $(m^2+1)/m^2 + (n^2+1)/n^2$ may be the square of an integer.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Let $m = b/a$, $n = d/c$, where a, b, c, d, are all integers, and where a is prime to b, and c to d. Substituting these values, $\dfrac{b^2+a^2}{b^2} + \dfrac{c^2+d^2}{d^2}$ must be the square of an integer. If it is to be an integer at all, clearly b must equal d, so that we have the equation

$$\frac{a^2+c^2+2b^2}{b^2} = \text{a perfect square},$$

or

$$a^2 + c^2 = b^2 \text{ [a perfect square } - 2] = b^2k,$$

where k must have one of the values 2 (*i. e.*, $2^2 - 2$), 7, 14, 23, 34, 47, 62, 79, 98, $\cdots$.

If k has one of the values 2, 34, 98, this equation possesses a solution, but not for any of the other values given. To show this and the method of obtaining the solution, it will be sufficient to treat the values $k = 2$, and $k = 7$. If $k = 7$, a and c must be prime to each other, and also to 7, as otherwise they would not be prime to b. Consequently each must be of the form $7\lambda \pm 1$, 2, or 3. But the sum of their squares cannot be divisible by 7 in that case, and the equation is impossible. If $k = 2$, let $x = a/b$ and $y = c/b$, so that the equation takes the form $x^2 + y^2 = 2$. An obvious solution is $x = 1$, $y = 1$. Any line through the point (1, 1) with rational slope will cut the circle $x^2 + y^2 = 2$ in a second rational point, and, conversely, taking the slope to be α/β, where α and β are integers, and solving for x and y, we get the general solution of our equation to be

$$x = \frac{a}{b} = \frac{\alpha^2 - 2\alpha\beta - \beta^2}{\alpha^2 + \beta^2}, \qquad y = \frac{c}{b} = \frac{\beta^2 - 2\alpha\beta - \alpha^2}{\alpha^2 + \beta^2},$$

where α and β are any integers. m and n are the reciprocals of x and y respectively.

A similar solution can readily be found in the other possible cases.

QUESTIONS AND DISCUSSIONS.

[Send all Communications to U. G. MITCHELL, University of Kansas, Lawrence, Kans.]

DISCUSSIONS.

I. Relating to Some Determinants Connected with the Bernoulli Numbers.

By K. P. Williams, Indiana University.

It is well known that the Bernoulli numbers B_1, B_2, B_3, $\cdots$ can be obtained from either of the following recurrence relations,

$$_{2n+1}C_{2n}B_n - {}_{2n+1}C_{2n-2}B_{n-1} + \cdots (-1)^{n-1}{}_{2n+1}C_2B_1 = (-1)^{n-1}(n - \tfrac{1}{2}),$$

or

$$_{2n+2}C_{2n}B_n - {}_{2n+2}C_{2n-2}B_{n-1} + \cdots (-1)^{n-1}{}_{2n+2}C_2B_1 = (-1)^{n-1}n,$$

where ${}_nC_r$ denotes the combinations of n things r at a time.[1] If we write out several of the equations we obtain the two series of relations

$$_3C_2B_1 = \tfrac{1}{2}, \qquad {}_4C_2B_1 = 1,$$

$$_5C_2B_1 - {}_5C_4B_2 = \tfrac{3}{2}, \qquad {}_6C_2B_1 - {}_6C_4B_2 = 2,$$

$$_7C_2B_1 - {}_7C_4B_2 + {}_7C_6B_3 = \tfrac{5}{2}, \qquad {}_8C_2B_1 - {}_8C_4B_2 + {}_8C_6B_3 = 3,$$

$$_9C_2B_1 - {}_9C_4B_2 + {}_9C_6B_3 - {}_9C_8B_4 = \tfrac{7}{2}, \qquad {}_{10}C_2B_1 - {}_{10}C_4B_2 + {}_{10}C_6B_3 - {}_{10}C_8B_4 = 4,$$

. . . .

$$\begin{vmatrix} {}_3C_2 & 0 & 0 & 0 & 1 \\ {}_5C_2 & {}_5C_4 & 0 & 0 & 3 \\ {}_7C_2 & {}_7C_4 & {}_7C_6 & 0 & 5 \\ {}_9C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 & 7 \\ {}_{10}C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 & 8 \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} {}_4C_2 & 0 & 0 & 0 & 2 \\ {}_6C_2 & {}_6C_4 & 0 & 0 & 4 \\ {}_8C_2 & {}_8C_4 & {}_8C_6 & 0 & 6 \\ {}_{10}C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 & 8 \\ {}_9C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 & 7 \end{vmatrix} = 0,$$

which show at once the scheme for writing out the corresponding determinants of order $n + 1$.

Let us consider the determinant on the left. Subtract three times the last column from the first. The first element in the first column becomes zero, and the other elements take the form

$$_rC_2 - 3(r-2) = \frac{r^2 - 7r + 12}{1 \cdot 2} = \frac{(r-3)(r-4)}{1 \cdot 2} = {}_{r-3}C_2.$$

When we expand with reference to the first row we obtain

$$\begin{vmatrix} {}_2C_2 & {}_5C_4 & 0 & 0 \\ {}_4C_2 & {}_7C_4 & {}_7C_6 & 0 \\ {}_6C_2 & {}_9C_4 & {}_9C_6 & {}_9C_8 \\ {}_7C_2 & {}_{10}C_4 & {}_{10}C_6 & {}_{10}C_8 \end{vmatrix} = 0.$$

A corresponding reduction of the other determinant can be made.

In the first determinant multiply the first four columns by 2, 4, 6, 8, respectively, and divide the rows by 3, 5, 7, 9, 10, respectively, and we obtain

$$\begin{vmatrix} {}_2C_1 & 0 & 0 & 0 & \frac{1}{3} \\ {}_4C_1 & {}_4C_3 & 0 & 0 & \frac{3}{5} \\ {}_6C_1 & {}_6C_3 & {}_6C_5 & 0 & \frac{5}{7} \\ {}_8C_1 & {}_8C_3 & {}_8C_5 & {}_8C_7 & \frac{7}{9} \\ {}_9C_1 & {}_9C_3 & {}_9C_5 & {}_9C_7 & \frac{8}{10} \end{vmatrix} = 0,$$

a similar transformation being possible for the second determinant.

Various other transformations of the original determinant are of course possible. A direct proof of the vanishing of some of these determinants might be of interest.

II. Relating to the Teaching of Logarithms.

By T. M. Simpson, University of Wisconsin.

The present note is the result of observations made in teaching logarithms to a large number of students. In departing somewhat from current textbook

practice, I have secured such uniformly good results that I feel there is merit in the method.

By way of introduction the class is drilled, if necessary, on the two laws of exponents which apply in logarithmic work:

$$10^m \times 10^n = 10^{m+n}, \text{ and } (10^m)^n = 10^{mn},$$

where the numbers, m and n, are positive or negative, integral or fractional, or zero.

The point of departure is now the equation,

$$10^L = A.$$

It is stated that this is also written, $L = \log A$, read "L is the logarithm of A." It is pointed out that A is necessarily positive. With no further preparation the class can now easily translate such forms as

$$10^2 = 100, \qquad \frac{10^2 \times 10^{-3}}{10^5} = 10^{2-3-5} = 10^{-6},$$

$$10^{-3} = .001, \qquad 10^{3.2} \times 10^{2.1} = 10^{5.3},$$

into the language of logarithms. Thus they get clearly in mind the concept of a logarithm as an exponent.

The proofs of the fundamental theorems may be treated as exercises in translating statements of exponent laws into logarithmic form. At first, however, the proofs are passed over rather lightly but the facts are insisted upon strongly.

Next, by a few concrete examples, it is shown that the logarithms of numbers having the same sequence of figures differ only by integers.

Take, for illustration, the equation, $10^L = 382$; multiply both members by 10, 10^2, 10^{-3}. The logarithm of 382 being L, the logarithms of 3820, 38200, .382, are seen to be, respectively, $L + 1$, $L + 2$, $L - 3$. Such examples as this lead directly to the observation that any number a can be written in the form $a' \times 10^n$, where a' is between 1 and 10, and n is a positive or negative integer or zero. This statement is not given to the student in the form written above. The situation is made clear to him and the idea is so simple that he may safely be left to formulate the statement for himself.

n is then the characteristic of the logarithm of a. No other rules for characteristics are given. This is a distinct gain over the usual method where there are four rules—two for direct, and two for inverse use of the tables. That the advantage is real is proved by the fact that students quickly see that the characteristic is given by the number of places they must move the decimal point from units' place to put the number in the form $a' \times 10^n$, and that they have little trouble in working both directly and inversely with the tables.

In computing with logarithms, I question the advantage of writing the characteristic -1 in the form $9. - 10$, and similarly for other negative characteristics. Why not say that if a characteristic is positive it can of course be

written before the mantissa, but if negative it must be placed after it? Thus, log .2 is .3010−1, log .02 is .3010−2, etc. In this way the actual characteristic appears with the logarithm, and labor in computation is somewhat cut down.

If it is required to divide .3010 − 1 by 3, change the logarithm to 2.3010 − 3. The student is made to see that he has an expression of the form $c + m - c'$ in which c and c' may be changed at pleasure, provided their difference remains constant. This leads naturally to the anticipation of another difficulty of the student, which is illustrated by the following problem. It is required to solve the equation, $.2 = .3^x$. The student easily obtains the result

$$x = \frac{\log .2}{\log .3} = \frac{9.3010 - 10}{9.4771 - 10},$$

and is unable to proceed. Perhaps I should say that three quarters of the class subtract the denominator from the numerator while, of the remainder, some know enough not to subtract but do not know what to do. My experience shows the curious fact that if the above result is written $\frac{.3010 - 1}{.4771 - 1}$ many more members of the class will see the correct procedure from this point. Most of the others will handle the fraction correctly if, just before the assignment of this type of problem, the instructor states that, when it is required to multiply or divide by a negative logarithm, the characteristic and mantissa are no longer separated (as always in addition or subtraction) but are combined. So if we wish to add or subtract log .2, we write it .3010 − 1, but if we wish to use it as a multiplier or divisor, we combine the parts and write it − .6990. This fact is seldom explicitly stated in the books and is such a stumbling block to the student that it seems wise to state it.

CORRESPONDENCE.

NOTE. A correspondent asks to be told something about descriptive geometry as a course of more than professional engineering concern. The Editors have asked Professor W. H. ROEVER, of Washington University, to frame a reply to this question, and we print it in full below since we believe it to be of interest to MONTHLY readers in general.

TO THE EDITORS OF THE MONTHLY:

In accordance with your request I submit the following reply to the question of your correspondent:

From my conversations and correspondence with various mathematicians in this country I have been led to the conclusion that many do not have a very clear notion of what descriptive geometry really is. Hence, I will first attempt to state precisely and concisely the nature and object of this branch of applied mathematics.

It is evident that drawing is done in a plane or on a surface. Hence graphical processes are executed in a plane (or on a surface). On the other hand the applied sciences frequently demand a graphical solution for the problems of space

of three (and of higher) dimensions. It, therefore, becomes necessary to represent the objects of space by means of figures in a plane and in such a way that the correspondence between the space object and its plane representative is unique and unambiguous, *i. e.*, it must be possible to pass from the space object to the plane representative and also to pass back again from the plane representative to the space object without ambiguity. If, therefore, we can find such plane representatives for the elements (points, lines, planes, etc.) of space, a problem of space is replaced by a problem in the plane. The plane problem may be solvable by graphical methods (*i. e.*, by a process of drawing) and then the solution of this plane problem is the plane representative of the solution of the given space problem. Hence, the space problem is solved by a graphical process.

Now, the object of descriptive geometry is to represent space objects by corresponding plane figures in such a way that the correspondence is unique and unambiguous and also to solve the problems of space by means of the solutions of the corresponding plane problems, the solutions of which are the plane representatives of the solutions of the given space problems.

In the remarks made above I have stressed the notion of graphical solutions of space problems. There are many branches of descriptive geometry which serve well this purpose. The most familiar branch is that known as the mongean method. By this method a point of space is represented by two points in the plane which lie on the same perpendicular to a line of the plane (called the ground line). A line of space is represented by two lines in the plane and a plane is represented by two lines which intersect on the ground line. As an example of how this method solves space problems let us consider the simple problem:
To find the line x of space which connects the two points A and B of space.

The point A of space is represented by the two points A', A'' of the plane and B is represented by B', B''. The line x' which connects A' and B' and the line x'' which connects A'' and B'', together form the pair of lines (x', x'') which represent the solution x of the given space problem. While the mongean method is a good method for the graphical solution of space problems, the plane representatives or pictures which it furnishes do not in general convey to the mind as adequate a notion of the space forms as do some other methods of descriptive geometry. For instance, the mongean representative (consisting of Figs. 1 and 2, page 268) of a certain bracket-shaped object, does not give as clear a notion of the space form of this object as does the picture in Fig. 3.

It thus appears that certain branches of descriptive geometry enable us to make pictures which help to develop the *power of space visualization*, and this power, as every mathematician knows, is a great asset, at least in some branches of mathematics.

An examination of the books on pure and applied mathematics shows clearly that the authors of mathematical works pay very little attention to the proper drawing of figures. Some persons take the point of view that figures are not necessary, but I think every one will admit that if figures are resorted to they should be properly constructed. In order that a figure such as Fig. 3 below

may serve to give the dimensions of the object which it represents as well as to convey to the mind an adequate notion of its form, it is essential to know the scales on the three axes $O'X'$, $O'Y'$, $O'Z'$ (which are the representatives of the three mutually perpendicular axes OX, OY, OZ, of space). If the figure is an orthographic projection on the plane of the paper, *either* the directions of the axes $O'X'$, $O'Y'$, $O'Z'$ may be chosen (*i. e.*, the angles $\xi = \sphericalangle Y'O'Z'$, $\eta = \sphericalangle Z'O'X'$,

Fig. 1.

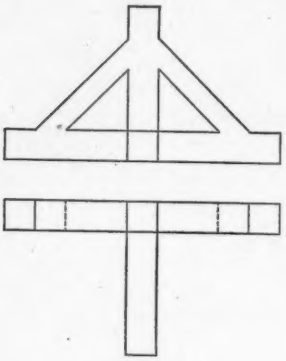


Fig. 2.

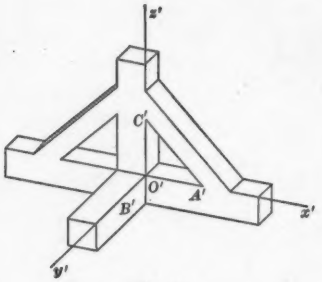


Fig. 3.

$\zeta = \sphericalangle X'O'Y'$ all greater than 90°), *or* the scales on these axes may be chosen ($l = O'A'$, $m = O'B'$, $n = O'C'$, $l^2 + m^2 > n^2$, $m^2 + n^2 > l^2$, $n^2 + l^2 > m^2$) but not *both*. The question then is, what is the relation between the axes and the scales? A theorem by Schwarz or a theorem by Gauss gives us information concerning this point. If, on the other hand, the projection is parallel and oblique the theorem of Pohlke tells us that both the angles and the scales may be chosen at random.

I believe that most mathematicians are not familiar with these fundamental theorems or with the fact that most of the representations which they attempt to make are a form of parallel projection known as Cavalier perspective, where $\sphericalangle X'O'Z' = 90°$. How properly to represent in figures of the type of Fig. 3 lines and planes which are perpendicular are also problems with which the average mathematician is not familiar.

Thus, it appears that if we wish adequately to represent and to plot space figures, some knowledge of *axonometry* (that branch of descriptive geometry which enables us to draw figures of the type of Fig. 3) is essential.

Hoping that these remarks will answer your question at least in part, I am

Sincerely yours,

WM. H. ROEVER.

WASHINGTON UNIVERSITY,
ST. LOUIS, MO.

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University, Bloomington, Ind.

At Dartmouth College, DR. F. J. MCMACKIN, of Columbia University, has been appointed instructor in mathematics.

Mr. R. W. DICKEY has been appointed associate professor of mathematics at Washington and Lee University.

At Vassar College, Dr. ELIZABETH B. COWLEY has been promoted from an assistant professorship to an associate professorship in mathematics.

Mr. E. F. CANADAY has been appointed to an instructorship in mathematics in the University of South Dakota for the coming year.

Mr. W. L. LORD becomes master in mathematics in Woodberry Forest School, Virginia, with the beginning of the new school year.

Mr. W. C. EELLS, who for the past three years has been instructor in mathematics at the U. S. Naval Academy, returns to his alma mater, Whitman College, as professor of applied mathematics. During the past summer he has been instructor in surveying at the Harvard University Engineering Camp at Ashland, New Hampshire.

At Yale University, Dr. G. I. TRACY has been promoted to an assistant professorship of mathematics.

Mr. F. S. NOWLAN, of Columbia University, has been appointed instructor in mathematics at the Carnegie School of Technology, Pittsburgh.

Professor ELLEN HAYES, of Wellesley College, will retire from active service at the end of the present academic year.

At Brown University, Mr. C. H. CURRIER has been promoted from instructor to an assistant professorship in mathematics.

Mr. J. H. HILL has been appointed professor of mathematics at Ohio Northern University.

Associate Professor L. C. EMMONS, of the Michigan Agricultural College, has been granted leave of absence for study at Harvard University. His place has been filled by the appointment of Mr. VERN JAMES of Indiana University.

In *Rendiconti della Accademia dei Lincei*, Rome, March, 1916, appears a paper by Professor ERNEST PASCAL, of the University of Rome, on "The mechanical solution of the general linear differential equation of the second order."

The Smith prizes, offered by the University of Cambridge for the year 1916, have been awarded as follows: To H. M. GARNER, of St. John's College, for his essay, "On orbital oscillation about the equatorial triangular configuration in the problem of three bodies"; to G. P. THOMPSON, of Corpus Christi College, for his essay, "On aëroplane problems."

The Rayleigh prize for 1916 has been awarded to W. M. SMART, of Trinity College, for his essay, "Libration in the Trojan planets."

A meeting of mathematicians from Sweden, Denmark, Finland and Norway was to be held at Stockholm from August 30 to September 2. It will be recalled that the sixth International Congress of Mathematicians was to have been held there at this time, and the fact that such a meeting was impossible doubtless led to this reunion.

A copy of De Morgan's "Differential and Integral Calculus" 1842, about 800 pages, is offered for sale by Mr. A. C. ANDREWS, Manual Training High School, Kansas City, Mo.

In the announcement concerning the Quarter-Centennial conferences at the University of Chicago printed in the June issue, mention should have been made of a report by Professor J. A. MILLER, Director of the Observatory at Swarthmore College, on the topic: "Choice of fields of comparison stars in parallax determinations."

The petition for the establishment of a section made by the members of the Association in the state of Iowa has been granted through the committee of the Council which has in charge the authorizing of sections. The Iowa Section is, therefore, duly constituted as the fourth section of the Mathematical Association of America.

Volume I, No. 3, April, 1916, of "The Texas Mathematics Teachers' Bulletin" published by the University of Texas, contains a number of articles of interest to teachers of elementary mathematics, among which may be enumerated: "Calculation with logarithms," by Professor M. B. PORTER; "Literal arithmetic," by Professor C. D. RICE; "What great men say about mathematics," "Is mathematics worth while in the high school," by Professor C. N. MOORE; "The mathematics of investment," by Professor E. L. DODD.

There has recently been issued in pamphlet form an extract from the last will and testament of Professor MITTAG-LEFFLER and his wife, this will having been made on March 16, 1916, Professor Mittag-Leffler's seventieth birthday. It seems that by the terms of this will the testators have established a Mathematical Institute to bear the name of the donors. This Institute will be housed in the villa now occupied by Professor Mittag-Leffler at Djursholm, Stockholm. The object is to establish a Foundation which shall assist in the development of pure mathematics in the four Scandinavian countries of Sweden, Denmark, Finland, and Norway, but more especially of Sweden. The terms of the will provide for awarding financial aid to students who give particular promise in the field of pure mathematics, and also to award medals and prizes for noteworthy achievements. So far as possible the prizes will consist of a gold medal, and of sets of the *Acta Mathematica*. These will be bestowed personally at the Institute, and will be the occasion of a formal ceremony. There has been issued also a sumptuous volume giving a complete catalogue of the magnificent library

of Professor Mittag-Leffler, which library will be deposited in the Institute and become available for students from all parts of the world. There has appeared hardly anything so sumptuous in the way of a mathematical bibliography, and the volume will be sought for by all mathematical bibliophiles. The founding of this Institute is one of the most noteworthy acts for the encouragement of mathematics that has yet been recorded in the history of the subject.

Dr. EMORY MCCLINTOCK, president of the New York Mathematical Society, 1890–1892, and president of the American Mathematical Society during the first two years of its existence, 1892–1894, died on July 10, 1916, at his home in Bay Head, N. J. Dr. McClintock was seventy-six years old at the time of his death. He was graduated from Columbia College in 1859; he was assistant professor of mathematics at Columbia from 1859 to 1860; he served as consular agent at Bradford, England, 1863–1866. His life work began in 1867, when he became Actuary of the Asbury Life Insurance Company, a position which he filled for four years; he then became actuary for the Northwestern Mutual Life Insurance Company, in which position he remained for seventeen years; he then became actuary and trustee for the Mutual Life Insurance Company of New York, positions which he held until his retirement from active service in 1911. Dr. McClintock was an honorary fellow of the American Academy of Arts and Sciences, a fellow of the Institute of Actuaries, London, a member and past president of the American Mathematical Society, and a charter member of the newly organized Mathematical Association of America. He made numerous contributions to mathematical periodicals, and during his long and active career was very much interested in the advancement of higher mathematics.

Professor W. H. METZLER, of Syracuse University, calls attention to the fact that the statement made in an article by Professor E. R. HEDRICK, in School and Society, March 11, 1916, would seem to him to lead to the impression that the scope of the Mathematics Teacher and the scope of the Association of Teachers of Mathematics of the Middle States and Maryland are restricted to the secondary field. It appears that while the actual activities of this organization and of its official journal have been, until the present year, almost exclusively in the secondary field, yet neither one has ever fixed any limits to its scope. Professor Hedrick requests that this means be taken to correct any misimpression which may have arisen from his article or from any other published statements in the MONTHLY or elsewhere in this connection.

In connection with the formation of sections of the Mathematical Association of America it will be of interest and value to take note of the geographical distribution of the charter membership, which is shown by the preliminary list published in the April number of the MONTHLY. An examination of that list shows that New York State has 120 members; the New England States have 125 members; the Middle States and Maryland, aside from New York, 134. The total for the North Atlantic States is 379; for the South Atlantic and other Southern States, aside from Missouri, 130; for the Far Western States, 102. Or,

making the summation in a slightly different way, in all the Atlantic States there are 429; in the Middle West including Missouri, 415 and in the remaining portions of the United States, 182; total for the United States, 1026; for foreign countries, 20; total individual members, 1046. Total institutional members, 52. Grand total, 1098.

Professor F. E. MILLER, of Otterbein University, Westerville, Ohio, desires to secure a copy of the October, 1913, issue of the MONTHLY. He will be glad to purchase it or to exchange for it a copy of the November, 1913, issue, of which he has two numbers. With this single exception he has a complete file of the MONTHLY from its beginning. This request is an example of recent activity on the part of very many in completing MONTHLY files. Any one who may have extra copies will confer a favor by reporting the same to the Managing Editor.

As this issue goes to press the first summer meeting of the Mathematical Association of America is about to assemble in the new buildings of the Massachusetts Institute of Technology. The preliminary announcement of the Committee on program shows that great care has been exercised in selecting the topics and speakers. Every omen seems to indicate an important series of papers on timely topics such as "The teaching of elementary dynamics," by Professor EDWARD V. HUNTINGTON of Harvard University; "The history of mathematical recreations," by Professor DAVID EUGENE SMITH of Columbia University; "Combined courses in mathematics for college freshmen," by Professor JOHN N. VAN DER VRIES of the University of Kansas; and "Combined courses in mathematics for freshmen in schools of technology," by Professor F. S. WOODS of the Massachusetts Institute of Technology.

It is understood that speakers were invited to lead in the formal discussion of these topics and that ample opportunity would be provided for informal discussion. Provision was also made for meetings of the Council, a meeting of Institutional Delegates, and for reports of committees, two or three of which had promised at least preliminary announcements of plans and progress.

On the next following pages are reprinted the Constitution and By-Laws of the Mathematical Association of America, together with the list of officers and members of the Council for 1916. Special attention is called to Article III of the Constitution and Section 2 of the By-Laws (concerning officers, tenure of office, and election of officers) by way of preparation for the first application of these regulations to be made during the autumn. The provision for nomination of officers through open primaries was intended to emphasize the opportunity presented to every member for active participation in the affairs of the Association.

CONSTITUTION AND BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

Article I—Name and Purpose.

1. This organization shall be known as The Mathematical Association of America.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

Article III—Officers.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad interim*.

Article IV—Meetings.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

Article V—Sections.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

Following are the Officers of the Association for 1916

For President, E. R. Hedrick, University of Missouri;

For Vice-Presidents, E. V. Huntington, Harvard University, and G. A. Miller, University of Illinois;

For Secretary-Treasurer, W. D. Cairns, Oberlin College;

For additional members of the Executive Council:

To serve for one year

D. N. Lehmer, University of California
R. E. Moritz, University of Washington
K. D. Swartzel, Ohio State University
Oswald Veblen, Princeton University

To serve for two years

R. C. Archibald, Brown University
Florian Cajori, Colorado College
M. B. Porter, University of Texas
J. W. Young, Dartmouth College

To serve for three years

B. F. Hinkel, Drury College
E. H. Moore, University of Chicago
J. N. Van der Vries, University of Kansas
Alexander Ziwet, University of Michigan

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Mathematics Lib.

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